Advanced Linear Algebra: Quiz 1 Solutions, Spring 2017

Problem 1. True or False:

(a) Any set containing a zero vector is linearly dependent.

Solution. **True**: $\mathbf{0}$ can always be written as a linear combination of any other vectors. ☐

(b) A basis must contain $\mathbf{0}$.

Solution. **False**. In fact, a basis can never contain $\mathbf{0}$ for the above reason. ☐

(c) Subsets of linearly dependent sets are linearly dependent.

Solution. **False**. For instance $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a linearly independent subset of the dependent set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$. ☐

(d) Subsets of linearly independent sets are linearly independent.

Solution. **True**. ☐

(e) If $\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$ then all the scalars α_k are zero.

Solution. **False** in general, e.g. if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are dependent. ☐

Problem 2. Assuming only the axioms for a vector space, prove that the zero vector $\mathbf{0}$ is unique. In other words, if $\mathbf{0}$ and $\mathbf{0}'$ satisfy $\mathbf{0} + \mathbf{v} = \mathbf{v}$ and $\mathbf{0}' + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$, then $\mathbf{0} = \mathbf{0}'$.

Solution. By the identity axiom, $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$, including the case $\mathbf{v} = \mathbf{0}'$, and likewise $\mathbf{0}' + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$ including $\mathbf{v} = \mathbf{0}$. Thus

$$\mathbf{0}' = \mathbf{0} + \mathbf{0}' = \mathbf{0}' + \mathbf{0} = \mathbf{0}.$$

☐

Problem 3. Assuming the axioms for a vector space, the identity $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v}$, and uniqueness of additive inverses (which both follow from the axioms), prove that $-\mathbf{v} = (-1)\mathbf{v}$, i.e., that the additive inverse of $\mathbf{v}$ is obtained by scalar multiplication by -1.

Solution. We have

$$\begin{aligned}\mathbf{0} &= 0\mathbf{v} \\ &= (1-1)\mathbf{v} \\ &= \mathbf{v} + (-1)\mathbf{v},\end{aligned}$$

so $(-1)\mathbf{v}$ is an additive inverse of $\mathbf{v}$, but this is unique. Thus $(-1)\mathbf{v} = -\mathbf{v}$. ☐