Advanced Linear Algebra: Quiz 2 Solutions, Spring 2017

Problem 1.

(a) Treating $\mathbb{C}$ as a complex vector space, show that multiplication by $\alpha = a + ib \in \mathbb{C}$ is a linear transformation from $\mathbb{C}$ to itself. What is its matrix?

Solution. We have $\alpha(z+w) = \alpha z + \alpha w$ and $\alpha(\omega z) = \omega\alpha z$ for all $z, w, \omega \in \mathbb{C}$, so complex multiplication is a linear transformation. The matrix with respect to the standard basis $\{\mathbf{e}_1 = 1\} \in \mathbb{C}$ is the 1×1 complex matrix α. □

(b) Treating $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, show that multiplication by $\alpha = a + ib \in \mathbb{C}$ defines a linear transformation on $\mathbb{R}^2$. What is its matrix?

Solution. Identifying $(x, y)^T \in \mathbb{R}^2$ with $x + iy \in \mathbb{C}$, we have

$$\alpha\big((x_1 + iy_1) + (x_2 + iy_2)\big) = \alpha(x_1 + iy_1) + \alpha(x_2 + iy_2),$$
$$\alpha\big(a(x + iy)\big) = a\alpha(x + iy), \quad \forall\, a \in \mathbb{R},$$

so α is linear. Evaluating α on the standard basis vectors, $\alpha(\mathbf{e}_1) \cong \alpha(1) = a + ib$ and $\alpha(\mathbf{e}_2) \cong \alpha(i) = -b + ia$, we conclude that multiplication by α is represented by the real 2×2 matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

□

Problem 2. Find the matrix of the orthogonal projection in $\mathbb{R}^2$ onto the line $x_2 = -\frac{1}{2}x_1$, by writing it as a composition of rotations and projection onto a coordinate axis.

Solution. The line makes an angle θ with the x_1-axis, such that $\sin\theta = 1/\sqrt{5}$ and $\cos\theta = 2/\sqrt{5}$. The projection P' is equivalent to the transformation which rotates the line onto the x_1-axis, projects onto the x_1-axis (P), and then rotates back: thus

$$\begin{aligned} P' &= R_{-\theta} P R_{\theta} \\ &= \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \\ &= \left(\frac{1}{\sqrt{5}}\right)^2 \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{pmatrix}. \end{aligned}$$

□

Problem 3. Let X be a subspace of a vector space V, and let $\mathbf{v} \in V$, $\mathbf{v} \notin X$. Prove that if $\mathbf{x} \in X$ then $\mathbf{x} + \mathbf{v} \notin X$.

Proof. Arguing by contradiction, suppose that $\mathbf{v} \in V$, $\mathbf{v} \notin X$, and $\mathbf{x} \in X$, but that $\mathbf{x} + \mathbf{v}$ lies in X. Since X is a subspace, it follows that $-\mathbf{x} = (-1)\mathbf{x} \in X$, and then

$$(\mathbf{x} + \mathbf{v}) + (-\mathbf{x}) = \mathbf{v} \in X,$$

which contradicts the assumption that $\mathbf{v} \notin X$. □