Chapter 1

The real number line

1.1 Ordered Sets

One basic property of many number systems (natural numbers, integers, rationals, etc) is that they are *ordered*, so we say that "3 *is greater than* 2", and so on.

1.1.1 Definition. A *total order* on a set S is a relation[1] $\leq$ satisfying the following axioms:

(O1) (Reflexivity) For every element a, it always holds that $a \leq a$.

(O2) (Antisymmetry) If $a \leq b$ and $b \leq a$, then it must be that $a = b$.

(O3) (Transitivity) If $a \leq b$ and $b \leq c$, then it holds that $a \leq c$.

(O4) (Totality) For every pair of elements a and b, either $a \leq b$ or $b \leq a$.

We say S is an *ordered set*.

1.1.2 Example. Find some examples of ordered sets.

1.1.3 Example. Find an example of a *partially ordered* set—a set with a relation satisfying axioms (O1)–(O3) but not (O4).

1.1.4 Problem. Suppose S is an ordered set. Formulate a reasonable definition of strict inequality ($a < b$) in terms of the order relation $\leq$. Then write down a definition equivalent to Definition 1.1.1 using strict inequality as the primitive relation; that is, write down a set of axioms that $<$ should satisfy, in terms of which $\leq$ (suitably defined in terms of $<$) has properties (O1)–(O4).

1.1.5 Definition. Let S be an ordered set, and $A \subseteq S$ a subset. An *upper bound* for A is an element $u \in S$ such that $a \leq u$ for every $a \in A$. If such an element exists, we say A is *bounded above*.

Similarly, a *lower bound* for A is an element $l \in S$ such that $l \leq a$ for every $a \in A$. If such a lower bound exists, we say A is *bounded below*.

1.1.6 Definition. A *least upper bound* or *supremum* of a bounded above set A is an element u_0 of S such that

(i) u_0 is an upper bound for A, and

(ii) $u_0 \leq u$ for every other upper bound u.

We denote a supremum for A (if it exists) by $\sup A$.

Similarly, a *greatest lower bound* or *infimum* of a bounded below set A is an element b_0 of S such that

(i) b_0 is a lower bound for A, and

[1] A relation is a comparision operation between two elements which evaluates to either *true* or *false*.

(ii) $b_0 \geq b$ for every other lower bound b.

We denote an infimum for A (if it exists) by $\inf A$.

1.1.7 Proposition. *If a supremum (or infimum) of A exists, then it is unique.*

1.1.8 Proposition. *If A and B are subsets of an ordered set S which are bounded above and below, and if $A \subseteq B$, then*

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

1.1.9 Example. Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ denote the set of integers, with the usual order. Find some examples of subsets A of $\mathbb{Z}$ such that

(a) A is bounded above and below.

(b) A is bounded above but not below.

(c) A is not bounded above and not bounded below.

Which of these sets have a supremum? Which have an infimum?

1.1.10 Example. Repeat Example 1.1.9 with the set $\mathbb{Q}$ of rational numbers in place of $\mathbb{Z}$. The following Lemma may be of use.

1.1.11 Lemma. *There exists no $q \in \mathbb{Q}$ such that $q^2 = 2$. [Possible hint: write $q = \frac{a}{b}$ in lowest terms and consider the evenness/oddness of a and b.]*

1.1.12 Definition. An ordered set S has the *least upper bound property* if every subset which is bounded above has a supremum. Likewise S has the *greatest lower bound property* if every subset which is bounded below has an infimum.

1.1.13 Example. Does $\mathbb{Z}$ have the least upper bound property? Does $\mathbb{Q}$? Justify your answers with a proof or counterexample.

1.1.14 Theorem. *If S has the least upper bound property, then it has the greatest lower bound property.*