Real Analysis Final Exam questions, Spring 2018

Problem 1. Prove the that uniform limit of continuous functions is continuous.

Problem 2. Show that every Riemann integrable function f on $[a,b] \subset \mathbb{R}$ is Lebesgue integrable and

$$\int_a^b f(x)\,dx = \int_{[a,b]} f\,d\lambda,$$

by the following steps.

(a) For an appropriate sequence of partitions (P_n) such that $U(f,P_n) - L(f,P_n) \to 0$, there are monotone sequences of simple functions ϕ_n and ψ_n corresponding to the lower and upper sums, respectively (so $\int_{[a,b]} \phi_n\,d\lambda = L(f,P_n)$ and $\int_{[a,b]} \psi_n\,d\lambda = U(f,P_n)$), such that $\phi_n \le f \le \psi_n$ for all n. Show that $\psi_n - \phi_n \to 0$ almost everywhere, and conclude that f is the almost everywhere limit of (ϕ_n).

(b) Show that f is measurable, and $\int_{[a,b]} f\,d\lambda = \lim_{n\to\infty} \int_{[a,b]} \phi_n\,d\lambda = \int_a^b f(x)\,dx$.

Problem 3 (Completion of a metric space)**.** Let X be an arbitrary metric space. Show that the map $i : X \longrightarrow \mathcal{C}(X;\mathbb{R})$, where

$$i(p) = f_p, \quad f_p(x) = d(x,p),$$

has the property that $\|i(p) - i(q)\| = d(p,q)$; in particular i is injective and continuous. Show that the closure $i(X)^- \subset \mathcal{C}(X;\mathbb{R})$ is a complete metric space in which X (or really the set $i(X)$ which we may identify with X) is dense.

Problem 4. Let K be a compact metric space and (f_k) a sequence of functions on K which is uniformly bounded and equicontinuous. For each $n \in \mathbb{N}$, define $g_n : K \longrightarrow \mathbb{R}$ by

$$g_n(x) = \max\{f_1(x), \dots, f_n(x)\}.$$

Show that the sequence (g_n) converges uniformly.

Problem 5 (Taylor's Theorem with integral remainder)**.**

(a) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a function which is $(k+1)$ times continuously differentiable. Prove that

$$g(1) = g(0) + g'(0) + \frac{1}{2!}g''(0) + \cdots + \frac{1}{k!}g^{(k)}(0) + \int_0^1 \frac{(1-t)^k}{k!}\frac{d^{k+1}}{dt^{k+1}}g(t)\,dt$$

(Hint: consider the last term and integrate by parts).

(b) Now let $f : A \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be $(k+1)$ times continuously differentiable (meaning all $(k+1)$-fold partial derivatives exist and are continuous)on a convex set A. Writing $x = (x_1, x_2)$ for points in A, prove that for all $x, y \in A$,

$$f(y) = \sum_{0\le k_1+k_2\le k} \frac{1}{k!}\frac{\partial^k f(x)}{\partial x_1^{k_1}\partial x_2^{k_2}}(y_1 - x_1)^{k_1}(y_2 - x_2)^{k_2} + \int_0^1 \frac{(1-t)^k}{k!}\frac{d^{k+1}}{dt^{k+1}} f\big((1-t)x + ty\big)\,dt.$$

Problem 6. Let $f \in \mathcal{C}([0,1])$ be a continuous function. Use Weierstrass approximation by polynomials to show that if

$$\int_0^1 x^n f(x)\,dx = 0$$

for all $n \ge 0$, then $f = 0$. (Hint: show that $\int_0^1 f(x)^2\,dx = 0$.)

Problem 7 (Implicit Function Theorem)**.** Let $F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a differentiable function. Denote points in the domain by (x, y), where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, and denote by $D_xF(x, y) \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $D_yF(x, y) \in L(\mathbb{R}^m, \mathbb{R}^m)$ the total derivative of F as a map $x \longmapsto F(x, y)$ (with y held constant) and as a map $y \longmapsto F(x, y)$ (with x held constant), respectively. Suppose that $F(x_0, y_0) = 0$ and $D_yF(x_0, y_0)$ is invertible. Assuming the Inverse Function Theorem, prove that there exists an open set $U \ni x_0$ and a unique function $g : U \longrightarrow \mathbb{R}^m$ such that $g(x_0) = y_0$ and

$$F(x, g(x)) = 0.$$

Show that

$$Dg(x_0) = -\big(D_yF(x_0, y_0)\big)^{-1} D_xF(x_0, y_0).$$

Problem 8 (Limits and derivatives under the integral sign)**.**

(a) Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \times (a, b) \longrightarrow \mathbb{R}$ a function such that $k_t(x) = f(x, t)$ is integrable for each $t \in (a, b)$ and $h_x(t) = f(x, t)$ is continuous for each $x \in X$. Suppose that there exists an integrable function g such that $|f(x, t)| \le g(x)$ for all t. Show that

$$\lim_{t\to t_0} \int_X f(x, t)\, d\mu = \int_X \lim_{t\to t_0} f(x, t)\, d\mu \quad \text{for every } t_0 \in (a, b).$$

In other words, $F(t) = \int_X f(x, t)\, d\mu$ is continuous in t. (Hint: recall that $h(t)$ is continuous if and only if $h(t_n) \to h(t)$ whenever $t_n \to t$; more generally $\lim_{t\to t_0} h(t) = L$ if and only if $h(t_n) \to L$ whenever $t_n \to t$.)

(b) Suppose now that $h_x(t) = f(x, t)$ is differentiable for each x and that there exists an integrable function $g(x)$ such that $\left|\frac{\partial}{\partial t} f(x, t)\right| \le g(x)$ for all t. Show that

$$\frac{d}{dt} \int_X f(x, t)\, d\mu = \int_X \frac{\partial}{\partial t} f(x, t)\, d\mu.$$

(Hint: Use the Mean Value Theorem.)

Problem 9. Denote by $GL(\mathbb{R}^n)$ the set of linear maps $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ which are invertible.

(a) Show that if $\|B\| < 1$, then the series $\sum_{n=0}^{\infty} B^n$ converges to $(I - B)^{-1}$.

(b) Define $\mathrm{Inv} : GL(\mathbb{R}^n) \subset L(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow L(\mathbb{R}^n, \mathbb{R}^n)$ by $\mathrm{Inv}(A) = A^{-1}$. Show that Inv is differentiable and $D\,\mathrm{Inv}(A)$ is the linear map defined by

$$D\,\mathrm{Inv}(A)B = -A^{-1}BA^{-1}.$$

Problem 10 (Completeness of L^1)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. For each integrable $f : X \longrightarrow \mathbb{R}$, define

$$\|f\|_{L^1} = \int_X |f|\, d\mu.$$

(a) Suppose that (f_k) is a sequence of integrable functions such that $\sum_{k=1}^{\infty} \|f_k\|_{L^1} < \infty$. Define $s = \sum_{k=1}^{\infty} |f_k| : X \longrightarrow [0, \infty]$ as a pointwise series (note that $s(x)$ may be $+\infty$ for some values of x). Use the Monotone Convergence Theorem to show that

$$\int_X s\, d\mu = \int_X \sum_{k=1}^{\infty} |f_k|\, d\mu = \sum_{k=1}^{\infty} \|f_k\|_{L^1}.$$

In particular, show that the set $E = \{x : s(x) = +\infty\}$ has measure zero.

(b) With (f_k) as above, show that $f = \sum_{k=1}^{\infty} f_k$ converges pointwise almost everywhere. Use the Dominated Convergence Theorem to show that

$$\int_X f\, d\mu = \int_X \sum_{k=1}^{\infty} f_k\, d\mu = \sum_{k=1}^{\infty} \int_X f_k\, d\mu.$$

(c) Suppose (g_k) is a sequence which is Cauchy in L^1 meaning for all $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$\|g_k - g_l\|_{L^1} < \varepsilon, \quad \text{for all } k, l \geq K.$$

Pass to a subsequence (g_{k_n}) such that $\left\|g_{k_{n+1}} - g_{k_n}\right\|_{L^1} < 2^{-n}$. Use the above to show that

$$\lim_{n\to\infty} g_{k_n} = \sum_{n=1}^{\infty} g_{k_{n+1}} - g_{k_n}$$

converges almost everywhere to an integrable function $g := \lim_n g_{k_n}$ and conclude that $\lim_k \|g_k - g\| = 0$, i.e., g_k converges to g in L^1.