DIFFERENTIAL EQUATIONS - TERMINOLOGY

An **ordinary differential equation** is an equation involving an unknown function of one variable $x(t)$ and its derivatives. The most general form of such an equation is

$$G\big(t, x, x', \dots, x^{(n)}\big) = 0$$

for some function G of $(n+2)$ variables, though in reasonably nice circumstances[1] we may rewrite this in **proper form**

$$x^{(n)} = F\big(t, x, x', \dots, x^{(n-1)}\big)$$

for some other function F. The highest order derivative which occurs in the equation (here n) is called the **order** of the differential equation. An nth order equation is always equivalent to a first order (n-dimensional) **system of equations**, wherein $x(t)$ is replaced by $X(t) = \big(x_1(t), \dots, x_n(t)\big)$, a function with vector values in $\mathbb{R}^n$. Following the book, we will write scalar valued functions in lower case and vector valued functions in upper case.

Thus we consider general equations of first order:

$$X' = F(t, X), \quad \text{or} \quad x' = F(t, x).$$

Such an equation is **autonomous** if F does not depend on t, i.e.

$$X' = F(X).$$

An equation is **linear** if F is linear in X, so

$$X' = A(t)X + B(t)$$

for $n \times n$ matrix valued $A(t)$ and $\mathbb{R}^n$ valued $B(t)$. Combining these two, a linear autonomous equation is therefore of the form

$$X' = A\,X$$

for a constant matrix A.

There are generally many solutions to a differential equation; under typical circumstances[2] an n-dimensional system has an n-parameter family of solutions called the **general solution**, and a unique **particular solution** satisfying each **initial condition**

$$X(0) = X_0, \quad X_0 \in \mathbb{R}^n.$$

For example, for an autonomous linear system, the general solution is of the form

$$X(t) = c_1 X_1(t) + \cdots + c_n X_n(t)$$

where $c_i \in \mathbb{R}$, $i = 1, \dots, n$, and $\{X_1(t), \dots, X_n(t)\}$ are linearly independent for all t, meaning no $X_i(t)$ can be written as a linear combination of the others.

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[1] To be precise, under the condition that the derivative of G with respect to its last variable is not 0, we may employ the Implicit Function Theorem of multivariable calculus to prove that such an F exists.

[2] Specifically, under the hypotheses of the Existence/Uniqueness Theorem for differential equations which we will prove later in the semester.

From now on we consider autonomous equations. Of particular importance are **equilibrium solutions**, which are solutions $X(t) =$ constant, and can be found by solving

$$F(X) = 0$$

since $X(t)$ is constant if and only if $X'(t) = 0$. An equilibrium solution is **stable** if all nearby solutions approach it asymptotically as $t \longrightarrow \infty$, and **unstable** if they diverge from it as $t \longrightarrow \infty$ (equivalently, if they approach it as $t \longrightarrow -\infty$).

We say a family of differential equations $X' = F(X, a)$ depending on some parameter a undergoes a **bifurcation** for some value of a if the overall structure of equilibria and/or their stability changes as a crosses this value.