DIFFERENTIAL EQUATIONS - TERMINOLOGY

An ordinary differential equation is an equation involving an unknown function of one variable \( x(t) \) and its derivatives. The most general form of such an equation is

\[
G(t, x, x', x^{(n)}) = 0
\]

for some function \( G \) of \((n + 2)\) variables, though in reasonably nice circumstances\(^1\) we may rewrite this in proper form

\[
x^{(n)} = F(t, x, x', \ldots, x^{(n-1)})
\]

for some other function \( F \). The highest order derivative which occurs in the equation (here \( n \)) is called the order of the differential equation. An \( n \)th order equation is always equivalent to a first order (\( n \)-dimensional) system of equations, wherein \( x(t) \) is replaced by \( X(t) = (x_1(t), \ldots, x_n(t)) \), a function with vector values in \( \mathbb{R}^n \). Following the book, we will write scalar valued functions in lower case and vector valued functions in upper case.

Thus we consider general equations of first order:

\[
X' = F(t, X), \quad \text{or} \quad x' = F(t, x).
\]

Such an equation is autonomous if \( F \) does not depend on \( t \), i.e.

\[
X' = F(X).
\]

An equation is linear if \( F \) is linear in \( X \), so

\[
X' = A(t)X + B(t)
\]

for \( n \times n \) matrix valued \( A(t) \) and \( \mathbb{R}^n \) valued \( B(t) \). Combining these two, a linear autonomous equation is therefore of the form

\[
X' = AX
\]

for a constant matrix \( A \).

There are generally many solutions to a differential equation; under typical circumstances\(^2\) an \( n \)-dimensional system has an \( n \)-parameter family of solutions called the general solution, and a unique particular solution satisfying each initial condition

\[
X(0) = X_0, \quad X_0 \in \mathbb{R}^n.
\]

For example, for an autonomous linear system, the general solution is of the form

\[
X(t) = c_1X_1(t) + \cdots + c_nX_n(t)
\]

where \( c_i \in \mathbb{R}, \ i = 1, \ldots, n \), and \( \{X_1(t), \ldots, X_n(t)\} \) are linearly independent for all \( t \), meaning no \( X_i(t) \) can be written as a linear combination of the others.

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\(^1\)To be precise, under the condition that the derivative of \( G \) with respect to its last variable is not 0, we may employ the Implicit Function Theorem of multivariable calculus to prove that such an \( F \) exists.

\(^2\)Specifically, under the hypotheses of the Existence/Uniqueness Theorem for differential equations which we will prove later in the semester.
From now on we consider autonomous equations. Of particular importance are **equilibrium solutions**, which are solutions $X(t) = \text{constant}$, and can be found by solving

$$F(X) = 0$$

since $X(t)$ is constant if and only if $X'(t) = 0$. An equilibrium solution is **stable** if all nearby solutions approach it asymptotically as $t \to \infty$, and **unstable** if they diverge from it as $t \to \infty$ (equivalently, if they approach it as $t \to -\infty$).

We say a family of differential equations $X' = F(X, a)$ depending on some parameter $a$ undergoes a **bifurcation** for some value of $a$ if the overall structure of equilibria and/or their stability changes as $a$ crosses this value.