**Math 1580 – Problem Set 9. Due Friday Nov. 18, 4pm**

Updated 11/13 to fix a typo in Problem 1. Thanks to Sarah for the catch.

**Problem 1.** Recall the following method of cofactor expansion to calculate the determinant of an \( n \times n \) matrix \( A \). Let \( A_{ij} \) denote the \((n-1) \times (n-1)\) matrix obtained from \( A \) by removing the \( i \)th row and \( j \)th column, and \( a_{ij} \) denote the \( i, j \)th entry of \( A \). Then \( \det(A) \) can be calculated by fixing a row, say row \( k \), and computing

\[
\det(A) = \sum_{j=1}^{n} (-1)^{j+k} a_{kj} \det(A_{kj})
\]

Similarly, we may fix a column instead, say column \( k \), and compute

\[
\det(A) = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det(A_{ik})
\]

Given \( A \) as above, define the cofactor matrix\(^1\) \( B \) to be the matrix whose \( i, j \) entry is

\[
b_{ij} = (-1)^{i+j} \det(A_{ji})
\]

(a) Prove that

\[
AB = BA = \det(A)I_n
\]

where \( I_n \) is the identity matrix. Conclude that provided \( \det(A) \neq 0 \), \( A^{-1} \) is given by

\[
A^{-1} = \frac{1}{\det(A)} B
\]

(b) Use this to prove that if \( A \) has integer entries and \( \det(A) = \pm 1 \), then \( A^{-1} \) has these same properties.

(c) Conclude that the \( n \times n \) matrices with integer entries and determinant \( \pm 1 \) form a group with respect to matrix multiplication, which we call \( \text{GL}(n, \mathbb{Z}) \).

**Problem 2.** Let \( L \) be a lattice in \( \mathbb{R}^n \), and suppose \( \dim(L) = n = \dim(\mathbb{R}^n) \). Show that a linearly independent set \( \{v_1, \ldots, v_n\} \subset L \) is a basis for \( L \) if and only if

\[
L \cap \mathcal{F}(v_1, \ldots, v_n) = 0
\]

where \( \mathcal{F}(v_1, \ldots, v_n) \) is the fundamental domain for \( \{v_1, \ldots, v_n\} \) defined as in class by

\[
\mathcal{F}(v_1, \ldots, v_n) = \{t_1v_1 + \cdots + t_nv_n : 0 \leq t_i < 1, \text{ for all } i\}
\]

Some hints:

(a) To show that (1) holds if \( v_1, \ldots, v_n \) is a basis, suppose that there is a vector \( v \in L \cap \mathcal{F}(v_1, \ldots, v_n) \) and show that \( v \) must be the zero vector.

(b) To show the other direction, let \( L' \) be the lattice generated by the \( v_i \), so that \( L' \subseteq L \). To show that \( L \subseteq L' \), let \( v \in L \) and write \( v \) as a linear combination (not necessarily with integer entries) of the \( v_i \), and use this to find a vector \( v' \in L' \) such that \( v - v' \in \mathcal{F}(v_1, \ldots, v_n) \). Conclude that \( v \) must equal \( v' \).

\(^1\)This is sometimes also called the “adjugate matrix.” Very unfortunately, it also sometimes called the “adjoint matrix,” which is a terrible practice since there is a different matrix obtained from \( A \) which is also called the adjoint and deserves the title much more.
Problem 3. Let \( L \subset \mathbb{R}^m \) be a lattice with basis \( \{v_1, \ldots, v_n\} \). We showed in class how to compute \( \det(L) \) as \( |\det(F(v_1, \ldots, v_n)| \) in the case that \( m = n \), where

\[
F(v_1, \ldots, v_n) = \begin{pmatrix}
| & \cdots & | \\
v_1 & \cdots & v_n \\
| & \cdots & |
\end{pmatrix}
\]

is the matrix whose columns consist of the components of the \( v_i \) as vectors in \( \mathbb{R}^m \).

This problem will give a way to compute this quantity even when \( m > n \). Note that in this case, the matrix (2) is still well-defined as a \( m \times n \) matrix.

(a) If \( v_1, \ldots, v_n \) are vectors in \( \mathbb{R}^m \), define the Gram matrix \( \text{Gram}(v_1, \ldots, v_n) \) to be the \( n \times n \) matrix whose \( i, j \) entry is the quantity

\[
[\text{Gram}(v_1, \ldots, v_n)]_{ij} = v_i \cdot v_j, \quad 1 \leq i, j \leq n.
\]

Show that

\[
\text{Gram}(v_1, \ldots, v_n) = F(v_1, \ldots, v_n)^TF(v_1, \ldots, v_n)
\]

(b) Show that if \( n = m \), then

\[
\det(\text{Gram}(v_1, \ldots, v_n)) = \det(L)^2.
\]  

(3)

(c) Show that if \( m > n \), then (3) still holds. Here are some hints:

(i) Argue that (3) holds if the \( v_i \) all lie in the subspace \( \{(x_1, \ldots, x_m) : x_{n+1} = \cdots = x_m = 0\} \subset \mathbb{R}^m \). We will reduce to this case below.

(ii) Remind yourself (or go learn!) that a (real valued) matrix is orthogonal if \( \det(R) = \pm 1 \), and that such matrices satisfy \( Rv_i \cdot Rv_j = v_i \cdot v_j \). Recall also that a matrix whose columns form a set of orthonormal vectors is an orthogonal matrix, and that orthogonal transformations preserve lengths, areas, volumes and so on. You may assume all these facts.

(iii) Enlarge the set \( \{v_1, \ldots, v_n\} \) to a basis \( \{v_1, \ldots, v_m\} \) for \( \mathbb{R}^m \) by adding \( m - n \) additional independent vectors. Let \( \{v_1^*, \ldots, v_m^*\} \) be an orthonormal set of vectors obtained from \( \{v_1, \ldots, v_m\} \) by the Gram-Schmidt procedure. Observe that the subspace spanned by the first \( n \) vectors in \( \{v_i^*\} \) is the same as that spanned by our original vectors \( \{v_1, \ldots, v_n\} \).

(iv) Form the orthogonal matrix \( R \) whose columns are the vectors \( v_1^*, \ldots, v_m^* \). Show that the linear transformation defined by \( R \) sends the subspace \( \{(x_{n+1} = \cdots = x_m = 0)\} \) to the space spanned by the vectors \( v_1, \ldots, v_n \). Argue that the inverse \( R^{-1} \) is also an orthogonal transformation, which does the reverse. Conclude the problem by showing that

\[
\text{Gram}(v_1, \ldots, v_n) = \text{Gram}(R^{-1}v_1, \ldots, R^{-1}v_n), \quad \text{and}
\]

\[
\text{Vol}_n(F(v_1, \ldots, v_n)) = \text{Vol}_n(F(R^{-1}v_1, \ldots, R^{-1}v_n))
\]

using your result from (cii) above.