Math 2420 Spring 2012 — Final Exam

Instructions:

- Due Wednesday 5/2 in class.
- Please complete the exam on your own — no collaboration.
- You may freely reference Hatcher and your notes from the class.
- Any other sources you consult need to be explicitly cited.

Problem 1. Using cup products, show that every map $S^{k+l} \longrightarrow S^k \times S^l$ induces the trivial homomorphism $H_{k+l}(S^{k+l};\mathbb{Z}) \longrightarrow H_{k+l}(S^k \times S^l;\mathbb{Z})$.

Problem 2. Suppose M is a closed oriented manifold of dimension $2k$. Show that if $H_{k-1}(M;\mathbb{Z})$ is torsion free then $H_k(M;\mathbb{Z})$ is also torsion free.

Problem 3. Given two abelian groups G and H, with associated Eilenberg-MacLane spaces $K(G,n)$ and $K(H,n)$, show that there is a bijection of sets

$$[K(G,n), K(H,n)] \cong \mathrm{Hom}(G,H)$$

where $[-,-]$ denotes homotopy classes of basepoint preserving maps.

Here you may have to use an important result that we did not cover in class. Namely, the *Hurewicz theorem* says that for an $(n-1)$-connected space X, the groups $\pi_n(X)$ and $H_n(X;\mathbb{Z})$ are isomorphic (at least for $n \geq 2$; if $n = 1$, then $H_1(X;\mathbb{Z})$ is the abelianization $\pi_1(X)/[\pi_1(X), \pi_1(X)]$).

Problem 4. Show that, on topological groups, the classifying space functor $B : G \longmapsto BG$ is a weak inverse to the (based) loopspace functor $\Omega : G \longmapsto \Omega G$ in the following sense:

(a) There is a weak equivalence $\Omega BG \longrightarrow G$.

(b) If G is path-connected, then there is a homotopy equivalence $B\Omega G \simeq G$.

(Things to consider: the pathspace fibration $PG \longrightarrow G$, fiber sequences.)

Problem 5. Show that every principal $\mathbb{R}^n$-bundle over a space B is trivial. You may assume B has the homotopy type of a CW complex. (Note that here we are considering $\mathbb{R}^n$ as an abelian group, not a vector space.)