Conservative Vector Fields

Theorem (Characterization of Conservative Vector Fields)**.** The following are equivalent for a vector field

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k} : R \subset \mathbb{R}^3 \to \mathbb{R}^3,$$

with *simply connected* domain $R \subset \mathbb{R}^3$.

1. $\mathbf{F}(x,y,z)$ is *conservative*; by definition
$$\mathbf{F} = \nabla f$$
for some scalar function (called a *potential function*) $f : R \subset \mathbb{R}^3 \to \mathbb{R}$.
2. Line integrals between two points are *path independent*:
$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}$$
for any two curves $\mathcal{C}_1$, $\mathcal{C}_2$ with the same starting and ending points.
3. Line integrals over closed curves vanish:
$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 0.$$
4. $\mathbf{F}$ is curl free:
$$\nabla \times \mathbf{F} = 0 \quad \text{on } R.$$
Here it is important that R is simply connected.

Proof. In order to show that any of 1)—4) imply the other three, we will prove that

$$4) \implies 3) \implies 2) \implies 1) \implies 4).$$

First the proof that 4) $\implies$ 3). Let $\mathcal{C}$ be a closed curve in R. Since R is simply connected, $\mathcal{C}$ can be contracted down to a point without leaving R. This defines a surface $\mathcal{S}$ (the one swept out by $\mathcal{C}$ as it is being contracted) such that $\mathcal{S} \subset R$ and $\partial\mathcal{S} = \mathcal{C}$. Since $\nabla \times \mathbf{F} = 0$ in R, and $\mathcal{S} \subset R$, we must have $\nabla \times \mathbf{F} = 0$ on $\mathcal{S}$, and therefore by Stokes' Theorem,

$$\oint_{\mathcal{C}=\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} (\nabla \times F) \cdot \mathbf{n}\, dS = 0.$$

To show that 3) $\implies$ 2), suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are curves with the same starting and ending points $\mathbf{p}_0$ and $\mathbf{p}_1$. Define a new curve $\mathcal{C}$ which follows $\mathcal{C}_1$ from $\mathbf{p}_0$ to $\mathbf{p}_1$, and then $\mathcal{C}_2$ in the reverse direction from $\mathbf{p}_1$ back to $\mathbf{p}_0$. Thus

$$\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2 \quad \text{is a closed curve.}$$

By 3),

$$0 = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}$$

so the line integrals over $\mathcal{C}_1$ and $\mathcal{C}_2$ must be equal.

Next we show 2) $\implies$ 1). We need to define a potential function f. First choose an arbitrary point $(x_0, y_0, z_0) \in R$. Next, to define the value of f at (x,y,z), let $\mathcal{C}$ be any curve from (x_0,y_0,z_0) to (x,y,z) and let

$$f(x,y,z) = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}.$$

By 3), it doesn't matter which curve we pick; $f(x, y, z)$ only depends on the point (x, y, z) and the point (x_0, y_0, z_0), so f is a well-defined scalar function. Note that if we had chosen a different point (x_1, y_1, z_1) instead of (x_0, y_0, z_0), we would have obtained a different function $g(x, y, z)$, but

$$f(x, y, z) - g(x, y, z) = \int_{\mathcal{C}'} \mathbf{F} \cdot d\mathbf{s} = c$$

where $\mathcal{C}'$ is some curve from (x_0, y_0, z_0) to (x_1, y_1, z_1). The right hand side is just a constant independent of (x, y, z), so our functions f and g would only differ by a constant, which is fine since potential functions are allowed to differ by a constant.

It remains to show that $\nabla f = \mathbf{F}$. Let $\mathcal{C}_x$ be the curve consisting of straight line segments from (x_0, y_0, z_0) to (x_0, y, z_0), then to (x_0, y, z), and finally to (x, y, z). Since $\mathcal{C}_x$ connects (x_0, y_0, z_0) to (x, y, z),

$$f(x, y, z) = \int_{\mathcal{C}_x} \mathbf{F} \cdot d\mathbf{s} = \int_{y_0}^{y} F_2(x_0, t, z_0)\, dt + \int_{z_0}^{z} F_3(x_0, y, t)\, dt + \int_{x_0}^{x} F_1(t, y, z)\, dt$$

where we have parametrized the three different segments of $\mathcal{C}_x$, and used the fact that $dx = dz = 0$ on the first, $dx = dy = 0$ on the second, and $dy = dz = 0$ on the third. Differentiating with respect to x and using the fundamental theorem of calculus, we find

$$\frac{\partial f}{\partial x}(x, y, z) = \frac{\partial}{\partial x}\left(\int_{y_0}^{y} F_2(x_0, t, z_0)\, dt + \int_{z_0}^{z} F_3(x_0, y, t)\, dt + \int_{x_0}^{x} F_1(t, y, z)\, dt\right) = F_1(x, y, z),$$

since the third term is the only place that x appears.

Similarly, letting $\mathcal{C}_y$ be the curve from (x_0, y_0, z_0) to (x, y_0, z_0) to (x, y_0, z) to (x, y, z), and letting $\mathcal{C}_z$ be the curve from (x_0, y_0, z_0) to (x, y_0, z_0) to (x, y, z_0) to (x, y, z), we find

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial}{\partial y}\left(\int_{\mathcal{C}_y} \mathbf{F} \cdot d\mathbf{s}\right) = F_2(x, y, z)$$

and

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{\partial}{\partial z}\left(\int_{\mathcal{C}_z} \mathbf{F} \cdot d\mathbf{s}\right) = F_3(x, y, z).$$

By the assumption 2), each of these curves is an equally valid choice to use for f, so it must be true that

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \mathbf{F}.$$

Finally to show that 1) $\implies$ 4), we just use the fact that

$$\nabla \times \mathbf{F} = \nabla \times (\nabla f) = 0$$

since the curl of a gradient is always zero. $\square$