**Math 350 Exam II Topics**

- Definition of multiple integrals over rectangular regions by Riemann sum approximations.
- Fubini's Theorem (not the proof), which allows us to compute multiple integrals as iterated single integrals. Valid for
  1. Continuous functions
  2. Bounded functions with mild discontinuities
  3. Unbounded functions as long as either $f \geq 0$ or $f \leq 0$ and one of the iterated integrals exist
- The change of variables theorem:

$$\iiint_{D'} f(x,y,z)\,dx\,dy\,dz = \iiint_{D} (f \circ \Phi)(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$
$$\text{where } \Phi : D \subset \mathbb{R}^3 \to D' \subset \mathbb{R}^3, \quad (u,v,w) \mapsto (x,y,z)$$

  and similarly for $\mathbb{R}^2$. Note that you can think of this in analogy to path integrals and surface integrals of scalar functions; we are using $(u,v,w)$ as a parametrization of the region $D'$.

- Special coordinate systems:

$$\begin{aligned}
\text{Polar:} &\quad (r,\theta) \mapsto (x,y) = (r\cos\theta, r\sin\theta), & dA &= r\,dr\,d\theta \\
\text{Cylindrical:} &\quad (r,\theta,z) \mapsto (x,y,z) = (r\cos\theta, r\sin\theta, z), & dV &= r\,dr\,d\theta\,dz \\
\text{Spherical:} &\quad (\rho,\phi,\theta) \mapsto (x,y,z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi), & dV &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.
\end{aligned}$$

- Path integrals (unoriented; for scalar functions):

$$\int_{\mathcal{C}} f(x,y,z)\,ds = \int_I (f \circ \mathbf{c})(t)\,\|\mathbf{c}'(t)\|\,dt$$
$$\text{for any parametrization } \mathbf{c} : I \subset \mathbb{R} \to \mathcal{C} \subset \mathbb{R}^3, \quad t \mapsto (x(t), y(t), z(t))$$

- Surface integrals (unoriented; for scalar functions):

$$\iint_{\mathcal{S}} f(x,y,z)\,dS = \iint_D (f \circ \Phi)(u,v)\,\|\mathbf{T}_u \times \mathbf{T}_v\|\,du\,dv$$
$$\text{for any parametrization } \Phi : D \subset \mathbb{R}^2 \to \mathcal{S} \subset \mathbb{R}^3, \quad (u,v) \mapsto (x(u,v), y(u,v), z(u,v))$$
$$\mathbf{T}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$
$$\mathbf{T}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

- Orientation for curves and surfaces
  - For curves, orientation is a choice of direction for unit tangent vector $\hat{\mathbf{T}}$.
  - An orientation preserving parametrization is a path $\mathbf{c}(t)$ such that $\mathbf{c}'(t)/\,\|\mathbf{c}'(t)\| = \hat{\mathbf{T}}$.
  - For surfaces, orientation is a choice of direction for unit normal vector $\hat{\mathbf{n}}$
  - An orientation preserving parametrization is $\Phi(u,v)$ such that $\mathbf{T}_u \times \mathbf{T}_v/\,\|\mathbf{T}_u \times \mathbf{T}_v\| = \hat{\mathbf{n}}$.

- Line integrals (for vector fields)

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{T}}\, ds = \pm \int_I (\mathbf{F} \circ \mathbf{c})(t) \cdot \mathbf{c}'(t)\, dt$$

(+ if $\mathbf{c}$ is orientation preserving, $-$ if $\mathbf{c}$ is orientation reversing.)

- (Oriented) surface integrals (for vector fields)

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \pm \iint_D (\mathbf{F} \circ \Phi)(u, v) \cdot \mathbf{T}_u \times \mathbf{T}_v\, du\, dv$$

(+ if $\Phi$ is orientation preserving, $-$ if $\Phi$ is orientation reversing.)

- Surface integral formulas for graphs $z = g(x, y)$:

$$\begin{aligned} d\mathbf{S} &= -g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k}\, dx\, dy \qquad \text{for } \hat{\mathbf{n}} \text{ pointing "up"}, \\ d\mathbf{S} &= g_x\mathbf{i} + g_y\mathbf{j} - \mathbf{k}\, dx\, dy \qquad \text{for } \hat{\mathbf{n}} \text{ pointing "down"}. \\ dS &= \sqrt{g_x^2 + g_y^2 + 1}dx\, dy \end{aligned}$$

- Conservative vector fields: $\mathbf{F}(x, y, z)$ such that

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z) \quad \text{for some } f$$

  - Fundamental Theorem of Calculus for Line Integrals. If $\mathbf{F}(x, y, z)$ is conservative, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$$

    where $(x_0, y_0, z_0)$ and $(x_1, y_1, z_1)$ are the starting and ending points of the (oriented) curve $\mathcal{C}$.

  - Therefore,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}'} \mathbf{F} \cdot d\mathbf{s}$$

    for any two curves $\mathcal{C}$ and $\mathcal{C}'$ which share the same starting and ending points.

  - Recovering $f$ from $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ if the latter is conservative:

    1. Set $f_x = F_1$, integrate (remember to include a general function $g(y, z)$ of the other two variables).
    2. Differentiate what you have so far obtained with respect to $y$, and compare this to $F_2$, which determines $g$ up to a general function $h(z)$.
    3. Differentiate what you have obtained with respect to $z$, and compare this to $F_3$, which determines $h$ up to a constant.

The following list of formulas will be printed on the first page of your exam:

$$d\mathbf{s} = \hat{\mathbf{T}}\,ds = \pm\mathbf{c}'(t)\,dt \qquad \text{+/- for orientation preserving/reversing}$$

$$d\mathbf{S} = \hat{\mathbf{n}}\,dS = \pm\mathbf{T}_u \times \mathbf{T}_v\,du\,dv \qquad \text{+/- for orientation preserving/reversing}$$

$$\mathbf{T}_u = x_u\mathbf{i} + y_u\mathbf{j} + z_u\mathbf{k}$$

$$\mathbf{T}_v = x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k}$$

$$d\mathbf{S} = \pm(-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k})\,dx\,dy \qquad \text{for a graph } z = g(x,y), \text{ +/- for } \hat{\mathbf{n}} \text{ up/down.}$$

$$ds = \|d\mathbf{s}\| = \|\mathbf{c}'(t)\|\ dt$$

$$dS = \|d\mathbf{S}\| = \|\mathbf{T}_u \times \mathbf{T}_v\|\ du\,dv$$

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} du\,dv\,dw$$

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\,dv$$

$$(x,y) = (r\cos\theta, r\sin\theta) \qquad dA = r\,dr\,d\theta$$

$$(x,y,z) = (r\cos\theta, r\sin\theta, z) \qquad dV = r\,dr\,d\theta\,dz$$

$$(x,y,z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta) \qquad dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$