Math 350 Problem Set 10 (due Friday 12/3 by 3pm)

Part I

1. (10pts) **Gauss' Law** The electric field due to a unit point charge at the origin is

$$\mathbf{E}(x,y,z) = \frac{1}{\rho^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$$

Let $\mathcal{S}$ be an arbitrary closed surface in $\mathbb{R}^3$. Prove that

$$\oiint_{\mathcal{S}} \mathbf{E}\cdot\hat{\mathbf{n}}\,dS = \begin{cases} 0 & \text{If } \mathcal{S} \text{ does not enclose } (0,0,0), \text{ and} \\ 4\pi & \text{if } \mathcal{S} \text{ encloses } (0,0,0). \end{cases}$$

(Hints: Calculate the divergence of $\mathbf{E}$, calculate the case in which $\mathcal{S}$ is a sphere of radius a centered at $(0,0,0)$, and use the Divergence Theorem judiciously.)

2. (15pts) There is an analogous theorem to the characterization of conservative vector fields that I proved in class for divergence free vector fields. In its most general form, it holds over regions $R \subset \mathbb{R}^3$ in which every closed surface can be contracted to a point without leaving R. However that version is quite difficult, and the result of deep mathematics. Here you will prove a simpler version, where $R = \mathbb{R}^3$.

Thus show that the following are equivalent (assume $\mathbf{F}$ is C^1).

(a) $\nabla\cdot\mathbf{F} = 0$ everywhere in $\mathbb{R}^3$

(b) For any closed surface $\mathcal{S} \subset \mathbb{R}^3$,

$$\oiint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 0.$$

(c) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are oriented surfaces such that $\partial\mathcal{S}_1 = \partial\mathcal{S}_2$ (with the same orientation), then

$$\iint_{\mathcal{S}_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{\mathcal{S}_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

(d) $\mathbf{F} = \nabla\times\mathbf{G}$ for a *vector field potential* (a.k.a. vector potential) $\mathbf{G}(x,y,z)$.

Show also that any two vector potentials $\mathbf{G}$ and $\mathbf{G}'$ must differ by a conservative vector field:

$$\mathbf{G}' - \mathbf{G} = \nabla h \qquad \text{for some } h.$$

(Suggestion: Show that (a) $\Longleftrightarrow$ (b), that (b) $\Longleftrightarrow$ (c), and that (a) $\Longleftrightarrow$ (d). Hints: in showing that (b) $\Longrightarrow$ (a), use the fact that if a continuous function f satisfies $\iiint_R f\,dV = 0$ for all $R \subset \mathbb{R}^3$, then $f = 0$. In showing that (a) $\Longrightarrow$ (d), try using $\mathbf{G}(x,y,z) = G_1(x,y,z)\mathbf{i} + G_2(x,y,z)\mathbf{j} + G_3(x,y,z)\mathbf{k}$, where

$$\begin{aligned} G_1(x,y,z) &= \int_0^z F_2(x,y,t)\,dt \\ G_2(x,y,z) &= -\int_0^z F_1(x,y,t)\,dt + \int_0^x F_3(t,y,0)\,dt \\ G_3(x,y,z) &= 0 \end{aligned}$$

You will have to use the fact that $\nabla\cdot\mathbf{F} = 0$. Of course this is not the only choice, as any other $\mathbf{G}$ which differs by a gradient field will do. However, this is probably the easiest.)

Part II

1. (5pts) Use Stokes' Theorem to calculate the work integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{F} = x \sin z\, \mathbf{i} + xy^2\, \mathbf{j} + y^2 \cos x\, \mathbf{k}$ and $\mathcal{C}$ is the unit circle in the x-y plane, oriented counterclockwise.

2. (5pts) For what constants a and b is the vector field

$$\mathbf{F} = (a \sin z + bxy^2)\mathbf{i} + 2x^2y\mathbf{j} + (x \cos z - z^2)\mathbf{k}$$

conservative? For these values of a and b, find a potential function f (so that $\mathbf{F} = \nabla f$).

3. (5pts) Verify the Divergence Theorem (i.e. calculate both sides and verify that they are equal)

$$\oiint_{\mathcal{S}=\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV$$

where $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} - z\mathbf{k}$, and V is the region bounded by $z = x^2 + y^2$ and the plane $z = 2$.

4. (5pts) Use the Divergence Theorem to determine the flux integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

where $\mathbf{F} = (x + yz^2)\mathbf{i} + x^2z\, \mathbf{j} + z\, \mathbf{k}$, and $\mathcal{S}$ is the upper ($z \geq 0$) unit hemisphere with upward pointing normal vector. (Note: $\mathcal{S}$ is not a closed surface.)

5. Let V be the tetrahedron (four sided figure) with vertices $P_0 = (0, 0, 0)$, $P_1 = (1, 0, 1)$, $P_2 = (1, 0, -1)$ and $P_3 = (1, 1, 0)$.

   (a) (2pts) For each of the four sides, give the orientation (in terms of order of the vertices) on the boundary curve of that side consistent with an outward pointing surface normal vector (pointing away from the tetrahedron).

   (b) (2pts) Compute directly the work integral

   $$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$$

   where $\mathbf{F} = yz\mathbf{j} - y^2\mathbf{k}$, and $\mathcal{C}$ is the boundary curve of the side $P_0P_1P_3$, with orientation as in part (a).

   (c) (2pts) Use Stokes' Theorem to compute the work done around the boundary curves of each of the four faces (including the one in part (b)), with orientations as in (a).

   (d) The sum of these four values should be 0. Explain this in two ways:

      i. (2pts) geometrically, by considering the various line integrals being added together, and

      ii. (2pts) by using the Divergence Theorem to compute the flux of $\nabla \times \mathbf{F}$ out of the tetrahedron.