Math 350 Problem Set 10 Solutions

Part I

1. (10pts) **Gauss' Law** The electric field due to a unit point charge at the origin is

$$\mathbf{E}(x,y,z) = \frac{1}{\rho^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$$

Let $\mathcal{S}$ be an arbitrary closed surface in $\mathbb{R}^3$. Prove that

$$\oiint_{\mathcal{S}} \mathbf{E}\cdot\hat{\mathbf{n}}\,dS = \begin{cases} 0 & \text{If } \mathcal{S} \text{ does not enclose } (0,0,0)\text{, and} \\ 4\pi & \text{if } \mathcal{S} \text{ encloses } (0,0,0). \end{cases}$$

(Hints: Calculate the divergence of $\mathbf{E}$, calculate the case in which $\mathcal{S}$ is a sphere of radius a centered at $(0,0,0)$, and use the Divergence Theorem judiciously.)

Solution. The divergence of $\mathbf{E}$ is (using the formulas $\frac{\partial}{\partial x}\rho = \rho/x$, $\frac{\partial}{\partial y}\rho = \rho/y$, etc)

$$\nabla\cdot\mathbf{E} = \left(\frac{1}{\rho^3} - 3\frac{x^2}{\rho^5} + \frac{1}{\rho^3} - 3\frac{y^2}{\rho^5} + \frac{1}{\rho^3} - 3\frac{z^2}{\rho^5}\right) = \frac{3\rho^2 - 3(x^2+y^2+z^2)}{\rho^5} = \begin{cases} 0 & \rho \neq 0 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Thus if $\mathcal{S}$ does not enclose $(0,0,0)$, then it is the boundary of a volumetric region V which doesn't contain $(0,0,0)$, and using the divergence theorem,

$$\oiint_{\mathcal{S}} \mathbf{E}\cdot\hat{\mathbf{n}}\,dS = \iiint_V \nabla\cdot\mathbf{E}\,dV = \iiint_V 0\,dV = 0 \quad \text{if } (0,0,0)\notin V.$$

If $\mathcal{S}$ *does* enclose the origin, we have to work harder. First let us calculate the surface integral explicitly for the sphere of radius a (with outward pointing normal). We parametrize using $(\theta,\phi)\in[0,2\pi]\times[0,\pi]$, where $dS = a^2\sin\phi\,d\phi\,d\theta$, and $\hat{\mathbf{n}} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/a$, to get

$$\begin{aligned}\oiint_{\mathcal{S}_a} \mathbf{E}\cdot\hat{\mathbf{n}}\,dS &= \iint_{\mathcal{S}_a} \frac{1}{a^3}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\frac{1}{a}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})dV \\ &= \int_0^{2\pi}\int_0^{\pi} \frac{1}{a^4}\underbrace{(x^2+y^2+z^2)}_{=a^2}a^2\sin\phi\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = 4\pi.\end{aligned}$$

Note that this answer is independent of a, so this is the answer for a sphere centered at $(0,0,0)$ of arbitrary radius.

Now, for a general surface $\mathcal{S}$ which encloses $(0,0,0)$, choose a sufficiently small a so that $\mathcal{S}_a$ fits entirely inside $\mathcal{S}$, and let V be the volumetric region between $\mathcal{S}$ and $\mathcal{S}_a$. Note that since V doesn't contain $(0,0,0)$, we know that $\nabla\cdot\mathbf{E} = 0$ in V. If the normal to both $\mathcal{S}$ and $\mathcal{S}_a$ is taken to be outward pointing, then

$$\partial V = \mathcal{S} - \mathcal{S}_a$$

so, by the divergence theorem,

$$\oiint_{\partial V} \mathbf{E}\cdot\hat{\mathbf{n}}\,dS = \oiint_{\mathcal{S}} \mathbf{E}\cdot\hat{\mathbf{n}}\,dS - \oiint_{\mathcal{S}_a} \mathbf{E}\cdot\hat{\mathbf{n}}\,dS = \iiint_V 0\,dV = 0.$$

Rearranging this, we get

$$\oiint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} \, dS = \oiint_{\mathcal{S}_a} \mathbf{E} \cdot \hat{\mathbf{n}} \, dS = 4\pi.$$

So we conclude that for an arbitrary closed surface $\mathcal{S}$ with outward normal,

$$\oiint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} \, dS = \begin{cases} 0 & \text{If } \mathcal{S} \text{ does not enclose } (0,0,0), \text{ and} \\ 4\pi & \text{if } \mathcal{S} \text{ encloses } (0,0,0). \end{cases}$$

2. (15pts) There is an analogous theorem to the characterization of conservative vector fields that I proved in class for divergence free vector fields. In its most general form, it holds over regions $R \subset \mathbb{R}^3$ in which every closed surface can be contracted to a point without leaving R. However that version is quite difficult, and the result of deep mathematics. Here you will prove a simpler version, where $R = \mathbb{R}^3$.

   Thus show that the following are equivalent (assume $\mathbf{F}$ is C^1).

   (a) $\nabla \cdot \mathbf{F} = 0$ everywhere in $\mathbb{R}^3$

   (b) For any closed surface $\mathcal{S} \subset \mathbb{R}^3$,
   $$\oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = 0.$$

   (c) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are oriented surfaces such that $\partial \mathcal{S}_1 = \partial \mathcal{S}_2$ (with the same orientation), then
   $$\iint_{\mathcal{S}_1} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iint_{\mathcal{S}_2} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS$$

   (d) $\mathbf{F} = \nabla \times \mathbf{G}$ for a *vector field potential* (a.k.a. vector potential) $\mathbf{G}(x, y, z)$.

   Show also that any two vector potentials $\mathbf{G}$ and $\mathbf{G}'$ must differ by a conservative vector field:

   $$\mathbf{G}' - \mathbf{G} = \nabla h \qquad \text{for some } h.$$

   (Suggestion: Show that (a) $\iff$ (b), that (b) $\iff$ (c), and that (a) $\iff$ (d). Hints: in showing that (b) $\implies$ (a), use the fact that if a continuous function f satisfies $\iiint_R f \, dV = 0$ for all $R \subset \mathbb{R}^3$, then $f = 0$. In showing that (a) $\implies$ (d), try using $\mathbf{G}(x, y, z) = G_1(x, y, z)\mathbf{i} + G_2(x, y, z)\mathbf{j} + G_3(x, y, z)\mathbf{k}$, where

   $$\begin{aligned} G_1(x, y, z) &= \int_0^z F_2(x, y, t) \, dt \\ G_2(x, y, z) &= -\int_0^z F_1(x, y, t) \, dt + \int_0^x F_3(t, y, 0) \, dt \\ G_3(x, y, z) &= 0 \end{aligned}$$

   You will have to use the fact that $\nabla \cdot \mathbf{F} = 0$. Of course this is not the only choice, as any other $\mathbf{G}$ which differs by a gradient field will do. However, this is probably the easiest.)

   Solution. First, we show (a) $\implies$ (b). So assume that $\nabla \cdot \mathbf{F} = 0$ everywhere. Let $\mathcal{S}$ be a closed surface, and suppose V is the region inside, so $\partial V = \mathcal{S}$. Then

   $$\oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iiint_V \nabla \cdot \mathbf{F} \, dV = 0$$

   which proves (b).

   Conversely, suppose (b) is true, so the surface integral of $\mathbf{F}$ vanishes for all closed surfaces. Then for an arbitrary region V,

   $$0 = \oiint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iiint_V \nabla \cdot \mathbf{F} \, dV$$

But if the integral of a function is 0 over every possible region, that function must vanish identically, so we conclude

$$\nabla \cdot \mathbf{F} = 0.$$

Now assume (b) is true, and we will prove (c). Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two surfaces with $\partial \mathcal{S}_1 = \partial S_2$. Thus they must have orientations which induce the same orientation on the curve $\partial \mathcal{S}_1 = \partial \mathcal{S}_2$, and we can construct a closed surface

$$\mathcal{S} = \mathcal{S}_1 - \mathcal{S}_2$$

which has a consistent orientation. But since surface integrals of $\mathbf{F}$ over closed surfaces are 0, we have

$$0 = \oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{\mathcal{S}_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS - \iint_{\mathcal{S}_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

so the two terms must be equal.

Assuming (c) is true, let $\mathcal{S}$ be a closed surface, and cut it along a closed curve two get two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ have the orientations from $\mathcal{S}$, then $\partial S_1 = -\partial S_2$, so in fact $\mathcal{S}_1$ and the surface $-\mathcal{S}_2$ have the same boundary. Thus

$$\oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{\mathcal{S}_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS + \iint_{\mathcal{S}_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{\mathcal{S}_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS - \iint_{-\mathcal{S}_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 0$$

since $\mathcal{S}_1$ and $-\mathcal{S}_2$ share a common boundary, which proves (b).

Assume (a) is true, and we will prove (d). So $\nabla \cdot \mathbf{F} = 0$ everywhere. Let $\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$ be the vector field from the hint:

$$\begin{aligned}
G_1(x,y,z) &= \int_0^z F_2(x,y,t)\, dt \\
G_2(x,y,z) &= -\int_0^z F_1(x,y,t)\, dt + \int_0^x F_3(t,y,0)\, dt \\
G_3(x,y,z) &= 0
\end{aligned}$$

Compute the curl of $\mathbf{G}$:

$$\nabla \times \mathbf{G} = \left(\frac{\partial}{\partial y}G_3 - \frac{\partial}{\partial z}G_2\right)\mathbf{i} + \left(\frac{\partial}{\partial z}G_1 - \frac{\partial}{\partial x}G_3\right)\mathbf{j} + \left(\frac{\partial}{\partial x}G_2 - \frac{\partial}{\partial y}G_1\right)\mathbf{k}$$

By the fundamental theorem of calculus,

$$\begin{aligned}
\frac{\partial G_2}{\partial z}(x,y,z &= -F_1(x,y,z) \\
\frac{\partial G_2}{\partial x}(x,y,z) &= -\int_0^z \frac{\partial}{\partial x}F_1(x,y,t)\, dt + F_3(x,y,0) \\
\frac{\partial G_1}{\partial z}(x,y,z) &= F_2(x,y,z) \\
\frac{\partial G_1}{\partial y}(x,y,z) &= \int_0^z \frac{\partial}{\partial y}F_2(x,y,t)\, dt
\end{aligned}$$

so

$$(\nabla \times \mathbf{G})(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + \left(F_3(x,y,0) - \int_0^z \left(\frac{\partial}{\partial x}F_2(x,y,t) + \frac{\partial}{\partial y}F_1(x,y,t)\right)\, dt\right)\mathbf{k}.$$

Since $\nabla \cdot \mathbf{F} = 0$, in particular at a point (x,y,t), we have

$$\frac{\partial}{\partial x}F_1(x,y,t) + \frac{\partial}{\partial y}F_2(x,y,t) = -\frac{\partial}{\partial z}F_3(x,y,t).$$

so the third term in the curl becomes

$$F_3(x,y,0) - \int_0^z \left(\frac{\partial}{\partial x} F_2(x,y,t) + \frac{\partial}{\partial y} F_1(x,y,t) \right) dt = F_3(x,y,0) + \int_0^z \frac{\partial}{\partial z} F_3(x,y,t)\, dt = F_3(x,y,z)$$

and thus

$$(\nabla \times \mathbf{G})(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k} = \mathbf{F}(x,y,z)$$

and (d) is proved.

Finally, showing that (d) $\implies$ (a) is easy, since if $\mathbf{F} = \nabla \times \mathbf{G}$,

$$\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{G}) = 0$$

as you showed on the last pset.

For the last claim, suppose $\mathbf{G}$ and $\mathbf{G}'$ are vector potentials for $\mathbf{F}$ so that

$$\mathbf{F} = \nabla \times \mathbf{G} = \nabla \times \mathbf{G}'.$$

Therefore

$$\nabla \times (\mathbf{G} - \mathbf{G}') = \mathbf{F} - \mathbf{F} = 0$$

everywhere, and so $\mathbf{G} - \mathbf{G}'$ is conservative:

$$\mathbf{G} - \mathbf{G}' = \nabla h, \quad \text{for some scalar function } h(x,y,z).$$

Part II

1. (5pts) Use Stokes' Theorem to calculate the work integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{F} = x \sin z\, \mathbf{i} + xy^2\, \mathbf{j} + y^2 \cos x\, \mathbf{k}$ and $\mathcal{C}$ is the unit circle in the x-y plane, oriented counterclockwise.

Solution. We compute

$$\nabla \times \mathbf{F} = 2y \cos x\, \mathbf{i} + \left(x \cos z + y^2 \sin x\right) \mathbf{j} + y^2\, \mathbf{k}.$$

We can choose $\mathcal{S}$ however we like, as long as $\partial\mathcal{S} = \mathcal{C}$. We see from the form of the curl of $\mathbf{F}$, that a good choice is to take $\mathcal{S}$ to be the unit disk in the x-y plane, since then we can take $\hat{\mathbf{n}} = \mathbf{k}$, and then $\mathbf{F} \cdot \hat{\mathbf{n}}\, dS = y^2\, dx\, dy$. Thus,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{\mathcal{S}} y^2\, dx\, dy = \int_0^{2\pi} \int_0^1 r^2 \sin^2\theta\, r\, dr\, d\theta = \frac{\pi}{4}.$$

2. (5pts) For what constants a and b is the vector field

$$\mathbf{F} = (a \sin z + bxy^2)\mathbf{i} + 2x^2y\mathbf{j} + (x \cos z - z^2)\mathbf{k}$$

conservative? For these values of a and b, find a potential function f (so that $\mathbf{F} = \nabla f$).

Solution. $\mathbf{F}$ will be conservative (on all of $\mathbb{R}^3$) if $\nabla \times \mathbf{F} = 0$ everywhere. We compute

$$\nabla \times \mathbf{F} = 0\, \mathbf{i} + (a \cos z - \cos z)\, \mathbf{j} + (4xy - 2bxy)\, \mathbf{k} = 0 \iff a = 1,\ b = 2.$$

Using these constants, we find a potential function f, first by requiring

$$f_x(x, y, z) = \sin z + 2xy^2 \implies f(x, y, z) = x \sin z + x^2y^2 + g(y, z).$$

Then

$$f_y(x, y, z) = 2x^2y + g_y(y, z) = 2x^2y \implies g_y(y, z) = 0 \implies g(y, z) = h(z).$$

Finally,

$$f_z(x, y, z) = x \cos z + h'(z) = x \cos z - z^2 \implies h'(z) = -z^2 \implies h(z) = -\frac{z^3}{3} + c$$

where we can take $c = 0$ if we like. Thus a potential function is given by

$$f(x, y, z) = x \sin z + x^2y^2 - \frac{z^3}{3}.$$

3. (5pts) Verify the Divergence Theorem (i.e. calculate both sides and verify that they are equal)

$$\oiint_{\mathcal{S}=\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV$$

where $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} - z\mathbf{k}$, and V is the region bounded by $z = x^2 + y^2$ and the plane $z = 2$.

Solution. The left hand side consists of two surface integrals. Let $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ where $\mathcal{S}_1$ is the surface $z = x^2 + y^2 : 0 \le z \le 2$ with downward pointing $\hat{\mathbf{n}}$ (so that $\hat{\mathbf{n}}$ points *away* from V), and $\mathcal{S}_1$ is the disk $z = 2 : x^2 + y^2 \le 2$ in the plane $z = 2$ with upward pointing $\hat{\mathbf{n}}$.

On $\mathcal{S}_1$ we have $\hat{\mathbf{n}}\, dS = (z_x\mathbf{i} + z_y\mathbf{j} - \mathbf{k})\ dx\, dy = (2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k})\ dx\, dy$ and we compute

$$\iint_{\mathcal{S}_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{D_{\sqrt{2}}} 4x^2 + 4y^2 + (x^2+y^2)\, dx\, dy = \int_0^{2\pi}\int_0^{\sqrt{2}} 5r^2\, r\, dr\, d\theta = 10\pi.$$

On $\mathcal{S}_2$ we have $\hat{\mathbf{n}}\, dS = \mathbf{k}\, dx\, dy$, $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} - 2\mathbf{k}$, and we compute

$$\iint_{\mathcal{S}_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{D_{\sqrt{2}}} -2\, dx\, dy = -4\pi.$$

Thus

$$\oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{\mathcal{S}_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS + \int_{\mathcal{S}_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 6\pi.$$

On the other hand,

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(-z) = 3$$

so

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{2} 3r\, dz\, dr\, d\theta = 2\pi \int_0^{\sqrt{2}} 3r\left(2 - r^2\right)\, dr = 6\pi.$$

4. (5pts) Use the Divergence Theorem to determine the flux integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

where $\mathbf{F} = (x + yz^2)\mathbf{i} + x^2z\,\mathbf{j} + z\,\mathbf{k}$, and $\mathcal{S}$ is the upper ($z \geq 0$) unit hemisphere with upward pointing normal vector. (Note: $\mathcal{S}$ is not a closed surface.)

Solution. We get a closed surface $\mathcal{S}'$ by taking $\mathcal{S}$ and adding the unit disk $D_1 = \{x^2 + y^2 \leq 1\}$ with *downward* pointing unit normal $\hat{\mathbf{n}} = \mathbf{k}$. Then $\mathcal{S}' = \partial V$ where V is the solid unit hemisphere. According to the Divergence Theorem,

$$\oiint_{\mathcal{S}'} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS + \iint_{D_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV.$$

Rearranging this, we have that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV - \iint_{D_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS.$$

The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x + yz^2) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(z) = 2.$$

so

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 2\left(\frac{1}{2}\frac{4}{3}\pi\right) = \frac{4}{3}\pi.$$

As for the other term, we have

$$\iint_{D_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_0^{2\pi}\int_0^1 -z\, dx\, dy = 0$$

since $z = 0$ on the x-y plane. Thus we conclude

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \frac{4}{3}\pi.$$

5. Let V be the tetrahedron (four sided figure) with vertices $P_0 = (0,0,0)$, $P_1 = (1,0,1)$, $P_2 = (1,0,-1)$ and $P_3 = (1,1,0)$.

(a) (2pts) For each of the four sides, give the orientation (in terms of order of the vertices) on the boundary curve of that side consistent with an outward pointing surface normal vector (pointing away from the tetrahedron).

Solution. These should be oriented by $P_0P_1P_3$, $P_0P_2P_1$, $P_1P_2P_3$, $P_0P_3P_2$, respectively (or any cyclic permutations thereof).

(b) (2pts) Compute directly the work integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{F} = yz\mathbf{j} - y^2\mathbf{k}$, and $\mathcal{C}$ is the boundary curve of the side $P_0P_1P_3$, with orientation as in part (a).

Solution. We break this up into three integrals over the curves $\mathcal{C}_1 = P_0P_1 : \{(x,y,z) = (t,0,t) \mid 0 \leq t \leq 1\}$, $\mathcal{C}_2 = P_1P_3 : \{(x,y,z) = (1,t,1-t) \mid 0 \leq t \leq 1\}$ and $\mathcal{C}_3 P_3P_0 : \{(x,y,z) = (-t,-t,0) \mid -1 \leq t \leq 0\}$. We find

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = 0$$

since $y = 0 \implies \mathbf{F} = 0$ on $\mathcal{C}_1$. On $\mathcal{C}_2$,

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 t - t^2 + t^2 \, dt = \frac{1}{2}.$$

On $\mathcal{C}_3$, we have, since $z = 0$ and $d\mathbf{s}$ has no $\mathbf{k}$ component

$$\int_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{s} = 0.$$

So

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2}.$$

(c) (2pts) Use Stokes' Theorem to compute the work done around the boundary curves of each of the four faces (including the one in part (b)), with orientations as in (a).

Solution. First off, we compute the curl of $\mathbf{F}$, obtaining

$$\nabla \times \mathbf{F} = -3y\mathbf{i}$$

Let us begin with the surface $\mathcal{S}_1 = P_0P_1P_3$. The equation of this plane is $z = x - y$, with upward normal, and so $\hat{\mathbf{n}}\, dS = (-\mathbf{i} + \mathbf{j} - \mathbf{k})\, dx\, dy$. The shadow region on the x-y plane is $R = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$. We compute

$$\iint_{\mathcal{S}_1} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_0^1 \int_0^x 3y \, dy \, dx = \frac{1}{2}.$$

which agrees with the above.

The surface $\mathcal{S}_2 = P_0P_3P_2$ lies in the plane $z = y - x$, and with $\hat{\mathbf{n}}$ pointing *down*, we have $\hat{\mathbf{n}}\, dS = (-\mathbf{i} + \mathbf{j} + \mathbf{k})\, dx\, dy$, with the same shadow region R as the previous surface. Since all that has changed is the sign on the $\mathbf{k}$ vector, and since $\nabla \times \mathbf{F}$ has no $\mathbf{k}$ component, the integral will be the same as that over $\mathcal{S}_1$:

$$\iint_{\mathcal{S}_2} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_0^1 \int_0^x 3y \, dy \, dx = \frac{1}{2}.$$

On $\mathcal{S}_3 = P_0P_2P_1$ which lies in the x-z plane, we have $\hat{\mathbf{n}} = -\mathbf{j}$, by inspection. Since $\nabla \times \mathbf{F}$ has no $\mathbf{j}$ component,

$$\iint_{\mathcal{S}_3} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 0$$

Finally, on $\mathcal{S}_4 = P_1P_2P_3$, which lies in the plane $x = 1$, we have $\hat{\mathbf{n}} = \mathbf{i}$ and $dS = dy\, dz$, with a shadow region $R' = \{(y, z) \mid y - 1 \leq z \leq 1 - y, 0 \leq y \leq 1\}$. Thus

$$\iint_{\mathcal{S}_3} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_0^1 \int_{y-1}^{1-y} -3y\, dz\, dy = \int_0^1 6y(y-1)\, dy = -1.$$

(d) The sum of these four values should be 0. Explain this in two ways:

i. (2pts) geometrically, by considering the various line integrals being added together, and
Solution. Given the orientations on the closed curves $\partial\mathcal{S}_i : i = 1, 2, 3, 4$, We see that the total work integral

$$\sum_{i=1}^{4} \oint_{\partial\mathcal{S}_i} \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{4} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

must vanish, since each line segment $P_iP_j : 0 \leq i < j \leq 4$ appears exactly twice, with opposite orientations.

ii. (2pts) by using the Divergence Theorem to compute the flux of $\nabla \times \mathbf{F}$ out of the tetrahedron.
Solution. The divergence of any curl is always zero, so

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

and therefore

$$\iiint_V \nabla \cdot (\nabla \times \mathbf{F})\, dV = 0$$

where V is the solid tetrahedron bounded by $\mathcal{S}_i : i = 1, 2, 3, 4$. But since we oriented $\mathcal{S}_i$ to point away from V, the Divergence Theorem gives

$$\sum_{i=1}^{4} \iint_{\mathcal{S}_i} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \oiint_{\partial V} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot (\nabla \times \mathbf{F})\, dV = 0.$$