Math 350 Problem Set 1 (Part I) (due Friday 9/10 by 3pm)

You may work on the problems in groups, but please write up your answers individually and cite your collaborators on the top of your assignment. If you consult sources other than the textbook to help you find answers to these questions, please cite these sources in your answer.
The following questions are worth 10 points each, and are the challenging ones. I will add more questions (Part II) this weekend, mostly from the textbook, which are more straightforward and computational, and which will have smaller point values. I will also post some optional problems (not to be graded) to work on if you think you need some more practice with the linear algebra we're using so far.

1. (10pts) Why is the following (incorrect) definition of the limit a bad one (i.e. why does is fail to express the idea that "f is close to $\mathbf{c}$ whenever $\mathbf{x}$ is close to $\mathbf{x}_0$")?

   Definition. $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{c}$ *if and only if, for all* $\delta > 0$, *there exists an* $\epsilon > 0$, *such that*

   $$\|\mathbf{x} - \mathbf{x}_0\| < \delta \implies \|f(\mathbf{x}) - \mathbf{c}\| < \epsilon.$$

2. In these two problems, either find the limit if it exists, and show your answer is correct by giving an ϵ-δ proof, or give an argument why the limit doesn't exist.

   (a) (10pts)
   $$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$$

   (b) (10pts)
   $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

3. The following theorems are proved in the internet supplement to the textbook. Try and prove them on your own using the definition of continuity and the ϵ-δ characterization of limits. If you're unable to prove them on your own, give yourself 5-10 minutes to look at the proof in the internet supplement, and then put it aside and try and reproduce the proof without consulting it again. Please cite the supplement if you end up using it.

   (a) (10pts) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{x} \in \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^l$ is continuous at $\mathbf{y} = f(\mathbf{x}) \in \mathbb{R}^m$, then $g \circ f : \mathbb{R}^n \to \mathbb{R}^l$ is continuous at $\mathbf{x}$. (Don't worry about domains and ranges; assume $g \circ f$ is defined.)

   (b) (10pts) If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuous at $\mathbf{x} \in \mathbb{R}^n$, then $g + f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{x}$. (Hint: you may be interested in using the triangle inequality $\|a + b\| \leq \|a\| + \|b\|$.)

4. (10pts) You're hiking on Mt. Badweather, whose height is described by a scalar function $h(x, y)$, where x and y represent latitude and longitude. All of a sudden (how could you have known?), a storm is upon you, and you need to get down fast. Describe your optimal route in terms of a curve $\mathbf{c} : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (x(t), y(t))$. Suppose you're only able to hike at a fixed speed ($\|\mathbf{c}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2} = 1$ for all t). Write the condition that $\mathbf{c}$ must satisfy in terms of h (and at each point in time) for you to get down as quickly as possible. Why is this the best choice?

5. (Extra credit: 10pts) In class, I showed an example (also in the textbook, p. 137, example 9) of a function $f : \mathbb{R}^2 \to \mathbb{R}$, both of whose partial derivatives exist at $(0, 0)$, but which is not continuous at $(0, 0)$. Can you find an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose partial derivatives exist *everywhere*, but is not continuous at some point? Why isn't this a counterexample to Theorem 8, p. 137, which says that if f is differentiable at $\mathbf{x}$, then f is continuous at $\mathbf{x}$? Try to come up with one on your own, and cite any sources if you find your example elsewhere.