Math 350 Problem Set 1 solutions

Part I

1. (10pts) Why is the following (incorrect) definition of the limit a bad one (i.e. why does is fail to express the idea that "f is close to $\mathbf{c}$ whenever $\mathbf{x}$ is close to $\mathbf{x}_0$")?

   Definition. $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{c}$ *if and only if, for all* $\delta > 0$, *there exists an* $\epsilon > 0$, *such that*

   $$\|\mathbf{x} - \mathbf{x}_0\| < \delta \implies \|f(\mathbf{x}) - \mathbf{c}\| < \epsilon.$$

   For any $\delta > 0$, I let $\epsilon = 1,000,000$, or even larger if I need. In this way, I can show that, according to this bad definition, $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ is equal to almost any $\mathbf{c}$ I want, regardless of whether or not f is actually approaching $\mathbf{c}$. This gives us an idea of why, in the proper definition, we require δ to exist based on the value of ϵ.

2. In these two problems, either find the limit if it exists, and show your answer is correct by giving an ϵ-δ proof, or give an argument why the limit doesn't exist.

   (a) (10pts)

   $$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$$

   By evaluating along some curves, we see that the limit probably exists, and is equal to 0. Indeed, a proof is as follows:

   Proof. Given $\epsilon > 0$, choose $\delta = \epsilon/2$. Then whenever

   $$\|\mathbf{x} - \mathbf{x}_0\| = \left(x^2 + y^2\right)^{1/2} < \delta,$$

   we have

   $$\begin{aligned} \|f(x,y) - 0\| &= \frac{\left|x^3 - y^3\right|}{x^2 + y^2} \leq \frac{|x|^3 + |y|^3}{x^2 + y^2} \\ &= \frac{(x^2)^{3/2} + (y^2)^{3/2}}{x^2 + y^2} \leq \frac{(x^2 + y^2)^{3/2} + (y^2 + x^2)^{3/2}}{x^2 + y^2} = 2(x^2 + y^2)^{1/2} < 2\delta = \epsilon. \end{aligned}$$

   □

   (b) (10pts)

   $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

   Along the coordinate axis $y = 0$, we have

   $$\lim_{x\to 0} \frac{x\,0}{x^2 + 0^2} = 0$$

   and similarly for $x = 0$. However, along $x = y$ we obtain

   $$\lim_{x\to 0} \frac{x^2}{x^2 + x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2} \neq 0.$$

   Therefore the limit as $(x, y) \to 0$ does not exist.

3. The following theorems are proved in the internet supplement to the textbook. Try and prove them on your own using the definition of continuity and the ϵ-δ characterization of limits. If you're unable to prove them on your own, give yourself 5-10 minutes to look at the proof in the internet supplement, and then put it aside and try and reproduce the proof without consulting it again. Please cite the supplement if you end up using it.

   (a) (10pts) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{x} \in \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^l$ is continuous at $\mathbf{y} = f(\mathbf{x}) \in \mathbb{R}^m$, then $g \circ f : \mathbb{R}^n \to \mathbb{R}^l$ is continuous at $\mathbf{x}$. (Don't worry about domains and ranges; assume $g \circ f$ is defined.)

   Proof. Let $\epsilon > 0$. By the continuity of g, there exists a $\delta_1 > 0$ such that

$$\|\mathbf{y} - \mathbf{y}_0\| < \delta_1 \implies \|g(\mathbf{y}) - g(\mathbf{y}_0)\| < \epsilon$$

   Let $\epsilon_2 = \delta_1$. By continuity of f, there exists a $\delta_2 > 0$ such that

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta_2 \implies \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \epsilon_2$$

   Set $\delta = \delta_2$. Thus, we have

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta \implies \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \delta_1 \implies \|g(f(\mathbf{x})) - g(f(\mathbf{x}_0))\| < \epsilon.$$

   □

   (b) (10pts) If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuous at $\mathbf{x} \in \mathbb{R}^n$, then $g + f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{x}$. (Hint: you may be interested in using the triangle inequality $\|a + b\| \leq \|a\| + \|b\|$.)

   Proof. Given $\epsilon > 0$, let $\epsilon_1 = \epsilon_2 = \epsilon/2$. By continuity of f and g, there exist δ_1 and δ_2 such that

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta_1 \implies \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \epsilon_1$$

   and

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta_2 \implies \|g(\mathbf{x}) - g(\mathbf{x}_0)\| < \epsilon_2.$$

   Choose δ to be the smaller of δ_1 and δ_2:

$$\delta = \min(\delta_1, \delta_2).$$

   Then

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta \implies \|f(\mathbf{x}) + g(\mathbf{x}) - f(\mathbf{x}_0) - g(\mathbf{x}_0)\| \leq \|f(\mathbf{x}) - f(\mathbf{x}_0)\| + \|g(\mathbf{x}) - g(\mathbf{x}_0)\| < \epsilon_1 + \epsilon_2 = \epsilon.$$

   □

4. (10pts) You're hiking on Mt. Badweather, whose height is described by a scalar function $h(x, y)$, where x and y represent latitude and longitude. All of a sudden (how could you have known?), a storm is upon you, and you need to get down fast. Describe your optimal route in terms of a curve $\mathbf{c} : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (x(t), y(t))$. Suppose you're only able to hike at a fixed speed ($\|\mathbf{c}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2} = 1$ for all t). Write the condition that $\mathbf{c}$ must satisfy in terms of h (and at each point in time) for you to get down as quickly as possible. Why is this the best choice?

   The problem is to write down a condition on $\mathbf{c}$ so that the composition $h \circ \mathbf{c} : \mathbb{R} \to \mathbb{R}$, which represents altitude as a function of time, is decreasing as rapidly as possible. By the chain rule,

$$\frac{d(h \circ \mathbf{c})}{dt}(t) = \nabla h(x(t), y(t)) \cdot \mathbf{c}'(t),$$

which we want to make as negative as possible. This occurs when $\mathbf{c}'(t)$ points exactly in the opposite direction as ∇h, since

$$\nabla h \cdot \mathbf{c}' = \|\nabla h\| \, \|\mathbf{c}'\| \cos\theta,$$

where θ is the angle between ∇h and $\mathbf{c}'$, and this expression is most negative for $\theta = \pi$. Since $\|\mathbf{c}'(t)\| = 1$, $\mathbf{c}'$ must be a unit vector in the opposite direction as ∇h, i.e.

$$\mathbf{c}'(t) = -\frac{\nabla h(x(t), y(t))}{\|\nabla h(x(t), y(t))\|}, \quad \text{for all } t. \tag{1}$$

That $\mathbf{c}(t)$ is uniquely determined by $\mathbf{c}(0)$ (our initial position) and equation (1) is the subject of a differential equations class, but it's true.

5. (Extra credit: 10pts) In class, I showed an example (also in the textbook, p. 137, example 9) of a function $f : \mathbb{R}^2 \to \mathbb{R}$, both of whose partial derivatives exist at $(0,0)$, but which is not continuous at $(0,0)$. Can you find an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose partial derivatives exist *everywhere*, but is not continuous at some point? Why isn't this a counterexample to Theorem 8, p. 137, which says that if f is differentiable at $\mathbf{x}$, then f is continuous at $\mathbf{x}$? Try to come up with one on your own, and cite any sources if you find your example elsewhere.

   In fact, a function on this homework is an example! (Provided we mend it just a bit). Indeed, let

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

   We showed above that this is not continuous at $(0,0)$, so it suffices to show that the partial derivatives exist everywhere. We can use ordinary calculus formulas to compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ for $(x,y) \neq (0,0)$:

$$\frac{\partial f}{\partial x} = \frac{y\left((x^2+y^2)^2 - 2x^2\right)}{(x^2+y^2)^4},$$

   and similarly,

$$\frac{\partial f}{\partial y} = \frac{x\left((x^2+y^2)^2 - 2y^2\right)}{(x^2+y^2)^4}.$$

   At $(x,y) = (0,0)$ we must compute using the definition of the partial derivative:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{\frac{h\,0}{h^2+0^2} - 0}{h} = 0$$

   and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \frac{\frac{0\,h}{0^2+h^2} - 0}{h} = 0.$$

   Of course, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not continuous at $(0,0)$; if they were, f would be continuously differentiable near $(0,0)$ and therefore continuous there.