Math 350 Problem Set 3 (Part I) (due Friday 9/24 by 3pm)

You may work on the problems in groups, but please write up your answers individually and cite your collaborators on the top of your assignment. If you consult sources other than the textbook to help you find answers to these questions, please cite these sources in your answer.

1. (5pts) Write the expression for the kth term in the Taylor series approximation of $f : \mathbb{R}^n \to \mathbb{R}$. You need not prove your answer, but think about why it is correct.

2. **Smooth versus analytic functions.** We've discussed the function classes $C^1(\mathbb{R}^n)$, $C^2(\mathbb{R}^n)$ and so on; the function class $C^k(\mathbb{R}^n)$ is the set

$$C^k(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \in C^0(\mathbb{R}^n) \text{ for all } 1 \leq i_1, \ldots, i_k \leq n \right\}.$$

Smooth functions, which we denote by the class $C^\infty(\mathbb{R}^n)$, are those which are in $C^k(\mathbb{R}^n)$ for *every* k. That is, f is smooth if every partial derivative of every order of f exists and is continuous. If f is smooth, we can write its Taylor approximation as an infinite series, since all derivatives exist:

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0) h_i + \cdots + R_\infty(\mathbf{x}_0, \mathbf{h})$$

where

$$\lim_{||h|| \to 0} \frac{R_\infty(\mathbf{x}_0, \mathbf{h})}{||h||^k} = 0 \quad \text{for all } k.$$

Analytic functions, denoted $C^\omega(\mathbb{R}^n)$ are smooth functions which are **equal** to their Taylor series; that is,

$$f \in C^\omega(\mathbb{R}^n) \iff R_\infty(\mathbf{x}_0, \mathbf{h}) = 0.$$

Of course we have $C^\omega(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \subset \cdots \subset C^2(\mathbb{R}^n) \subset C^1(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$.

(a) (10pts) Show that there are smooth functions in $C^\infty(\mathbb{R})$ which are not analytic, by showing that

$$f : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} e^{-1/t} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

is a counterexample. That is, show that all derivatives of $e^{-1/t}$ at $t = 0$ are equal to 0, so that the Taylor series of $e^{-1/t}$ at $t = 0$ is

$$\sum_{k=1}^{\infty} \frac{f^{(k)}(0) t^k}{k!} = 0$$

Nevertheless, $e^{-1/t} \neq 0$ for $t > 0$. What is the remainder term $R_\infty(0, t)$?

(Hint: Show that $\frac{d^k f(t)}{dt^k}$ has the form $p_k(t) \frac{e^{-1/t}}{t^{2k}}$ where $p_k(t)$ is some polynomial of order $k - 1$, which you need not calculate explicitly. You may use the fact that negative exponentials decay faster than any polynomial. Don't knock yourself out on this part.)

(b) (10pts) Produce an example of a function $f \in C^\infty(\mathbb{R}^n)$, $n > 1$ which is not analytic, and which depends explicitly on all variables (i.e. not just the function $(x_1, \ldots, x_n) \mapsto e^{-1/x_1}$.)

(Hint: Compose some reasonable function $g : \mathbb{R}^n \to \mathbb{R}$ with $e^{-1/t} : \mathbb{R} \to \mathbb{R}$ and argue convincingly using the chain rule.)

3. (15pts) By definition, the derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ at $\mathbf{x}_0 \in \mathbb{R}^n$ is the unique linear function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0.$$

Of course, we call this function $\mathbf{T} = \mathbf{D}f(\mathbf{x}_0)$. I asserted in lecture that $\mathbf{T}$ is given by matrix multiplication by the matrix of partial derivatives at $\mathbf{x}_0$:

$$\mathbf{T}\,\mathbf{v} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Prove this.

(Hint: Show that T_{ij} satisfies

$$\lim_{h\to 0} \frac{|f_i(x_1,\ldots,x_j+h,\ldots,x_n) - f_i(x_1,\ldots,x_n) - T_{ij}h|}{|h|} = 0$$

which is equivalent to the statement

$$T_{ij} = \lim_{h\to 0} \frac{f_i(x_1,\ldots,x_j+h,\ldots,x_n) - f_i(x_1,\ldots,x_j,\ldots,x_n)}{h} = \frac{\partial f_i}{\partial x_j}(x_1,\ldots,x_n)$$

where $\mathbf{x}_0 = (x_1,\ldots,x_n)$.)