Math 350 Problem Set 3 Part I Solutions

1. (5pts) Write the expression for the kth term in the Taylor series approximation of $f : \mathbb{R}^n \to \mathbb{R}$. You need not prove your answer, but think about why it is correct.

   In the approximation for $f(\mathbf{x}_0 + \mathbf{h})$ in terms of f and its derivatives at $\mathbf{x}_0$, the kth term is

   $$\frac{1}{k!} \sum_{i_1,\dots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(\mathbf{x}_0) h_{i_1} \cdots h_{i_k}.$$

   This follows from iterating the proof that we did in class to determine the first and second terms. Writing $g(t) = f(\mathbf{x}_0 + t\mathbf{h})$, the term is equal to $\frac{1}{k!} g^{(k)}(0)$. We determine $g^{(k)}(t)$ inductively. By the chain rule,

   $$g'(t) = \mathbf{D}f(\mathbf{x}_0 + t\mathbf{h}) \cdot \mathbf{h} = \sum_{i_1=1}^{n} \frac{\partial f}{\partial x_{i_1}}(\mathbf{x}_0 + t\mathbf{h}) h_{i_1}$$

   Then given that the $(k-1)$st derivative of g at t has the form

   $$g^{(k-1)}(t) = \sum_{i_1,\dots,i_{k-1}=1}^{n} \frac{\partial^{k-1} f}{\partial x_{i_1} \cdots \partial x_{i_{k-1}}}(\mathbf{x}_0) h_{i_1} \cdots h_{i_{k-1}},$$

   we differentiate each term, and use the chain rule to get

   $$g^{(k)}(t) = \sum_{i_1,\dots,i_{k-1}=1}^{n} \sum_{i_k=1}^{n} \frac{\partial}{\partial x_{i_k}} \left(\frac{\partial^{k-1} f}{\partial x_{i_1} \cdots \partial x_{i_{k-1}}}(\mathbf{x}_0) h_{i_1} \cdots h_{i_{k-1}} \right) h_{i_k}$$

   and the result then follows by collecting the summations, using symmetry of mixed partial derivatives, evaluating at $t = 0$ and putting the requisite $\frac{1}{k!}$ in front.

2. **Smooth versus analytic functions.** We've discussed the function classes $C^1(\mathbb{R}^n)$, $C^2(\mathbb{R}^n)$ and so on; the function class $C^k(\mathbb{R}^n)$ is the set

   $$C^k(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \in C^0(\mathbb{R}^n) \text{ for all } 1 \le i_1, \dots, i_k \le n \right\}.$$

   Smooth functions, which we denote by the class $C^\infty(\mathbb{R}^n)$, are those which are in $C^k(\mathbb{R}^n)$ for *every* k. That is, f is smooth if every partial derivative of every order of f exists and is continuous. If f is smooth, we can write its Taylor approximation as an infinite series, since all derivatives exist:

   $$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0) h_i + \cdots + R_\infty(\mathbf{x}_0, \mathbf{h})$$

   where

   $$\lim_{\|h\| \to 0} \frac{R_\infty(\mathbf{x}_0, \mathbf{h})}{\|h\|^k} = 0 \quad \text{for all } k.$$

   Analytic functions, denoted $C^\omega(\mathbb{R}^n)$ are smooth functions which are **equal** to their Taylor series; that is,

   $$f \in C^\omega(\mathbb{R}^n) \iff R_\infty(\mathbf{x}_0, \mathbf{h}) = 0.$$

   Of course we have $C^\omega(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \subset \cdots \subset C^2(\mathbb{R}^n) \subset C^1(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$.

(a) (10pts) Show that there are smooth functions in $C^\infty(\mathbb{R})$ which are not analytic, by showing that

$$f:\mathbb{R}\to\mathbb{R},\qquad t\mapsto\begin{cases} e^{-1/t} & \text{for } t>0\\ 0 & \text{for } t\le 0\end{cases}$$

is a counterexample. That is, show that all derivatives of $e^{-1/t}$ at $t=0$ are equal to 0, so that the Taylor series of $e^{-1/t}$ at $t=0$ is

$$\sum_{k=1}^{\infty}\frac{f^{(k)}(0)t^k}{k!}=0$$

Nevertheless, $e^{-1/t}\neq 0$ for $t>0$. What is the remainder term $R_\infty(0,t)$?
(Hint: Show that $\frac{d^kf(t)}{dt^k}$ has the form $p_k(t)\frac{e^{-1/t}}{t^{2k}}$ where $p_k(t)$ is some polynomial of order $k-1$, which you need not calculate explicitly. You may use the fact that negative exponentials decay faster than any polynomial. Don't knock yourself out on this part.)

Clearly the derivative from the left of $f(t)$ at $t=0$ is 0. So we need to show that

$$\lim_{t\to 0^+}\frac{d^k}{dt^k}\left(e^{-1/t}\right)=0\qquad\text{for all } k.$$

We'll prove the claim in the hint by induction. (You needn't have been this formal; taking the first two derivatives and then generalizing would be OK). The claim is true for $k=0$, namely that

$$e^{-1/t}=\frac{1\,e^{-1/t}}{t^{0k}}$$

and $1=p_1(t)$ is certainly a polynomial of order $0=1-1$. Now assume the claim is true for $k-1$, and we'll prove that it's true for k. So our assumption is that

$$\frac{d^{k-1}}{dt^{k-1}}\left(e^{-1/t}\right)=\frac{p_{k-1}(t)e^{-1/t}}{t^{2(k-1)}}$$

where $p_{k-1}(t)$ is a polynomial of order $(k-1)-1=k-2$. Let's differentiate this. We get

$$\frac{d^k}{dt^k}\left(e^{-1/t}\right)=\frac{p'_{k-1}(t)e^{-1/t}}{t^{2(k-1)}}+\frac{p_{k-1}(t)e^{-1/t}}{t^2\,t^{2(k-1)}}-2(k-1)\frac{p_{k-1}(t)e^{-1/t}}{t^{2(k-1)+1}},$$

differentiating $p_{k-1}(t)$, $e^{-1/t}$, and $t^{-2(k-1)}$, respectively. Putting everything over the common denominator t^{2k}, we get

$$\frac{d^k}{dt^k}\left(e^{-1/t}\right)=\frac{\left(t^2\,p'_{k-1}(t)+p_{k-1}(t)-2(k-1)t\,p_{k-1}(t)\right)e^{-1/t}}{t^{2k}}.$$

Consider the order of the polynomial multiplying $e^{-1/t}$. $p'_{k-1}(t)$ is of order $k-3$, since it's the derivative of a polynomial of order $k-2$, so $t^2\,p'_{k-1}(t)$ has order $k-3+2=k-1$. $p_{k-1}(t)$ has order $k-2$, so that's fine, and $t\,p_{k-1}(t)$ has order $k-2+1=k-1$. So

$$p_k(t):=\left(t^2p'_{k-1}(t)+(1-2(k-1)t)p_{k-1}(t)\right)$$

has order $k-1$ as claimed.
Now we need to show that

$$\lim_{t\to 0^+}\frac{p_k(t)e^{-1/t}}{t^{2k}}=0.$$

In fact it suffices to show that

$$\lim_{t\to 0^+} \frac{c\,e^{-1/t}}{t^{2k}} = 0.$$

since $|p_k(t)| \le c$ for some c when t is sufficiently small. You can use the fact that I gave you, namely that negative exponentials decay faster than any polynomial, obtaining

$$\lim_{s\to\infty} c\,s^{2k}\,e^{-s} = 0$$

where $s = 1/t$. To be completely rigorous, however, one would have to work a bit harder, using the power series formula for $e^{1/t}$:

$$c\,t^{-2k} = \frac{c\,(2k+1)!}{(2k+1)!t^{2k}} \le c\,(2k+1)! \sum_{n=0}^{\infty} \frac{1}{n!} t \left(\frac{1}{t}\right)^n = c\,(2k+1)!\,t\,e^{1/t}$$

so

$$\frac{c\,e^{-1/t}}{t^{2k}} \le c\,(2k+1)!\,t \to 0 \qquad \text{as } t \to 0.$$

(b) (10pts) Produce an example of a function $f \in C^\infty(\mathbb{R}^n)$, $n > 1$ which is not analytic, and which depends explicitly on all variables (i.e. not just the function $(x_1, \ldots, x_n) \mapsto e^{-1/x_1}$.)
(Hint: Compose some reasonable function $g : \mathbb{R}^n \to \mathbb{R}$ with $e^{-1/t} : \mathbb{R} \to \mathbb{R}$ and argue convincingly using the chain rule.)

Let $g(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$. So our function is

$$\mathbf{x} \mapsto \begin{cases} e^{-1/x_1+\cdots+x_n} & \text{if } x_i > 0 \text{ for all } i, \\ 0 & \text{otherwise} \end{cases}$$

Its derivative at $\mathbf{x}$ is

$$\mathbf{D}(f\circ g)(\mathbf{x}) = f'(g(\mathbf{x}))\mathbf{D}g(\mathbf{x}) = f'(g(\mathbf{x}))\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$$

by the chain rule. As $\mathbf{x} \to \mathbf{0}$, $\mathbf{D}g(\mathbf{x})$ is constant, while

$$f'(g(\mathbf{x}_0)) \to 0.$$

The second derivative at $\mathbf{x}$ is

$$\mathbf{D}^2(f\circ g)(\mathbf{x}) = f''(g(\mathbf{x}))\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

which again goes to 0. Higher derivatives are similar, except we can no longer use matrices to represent them.

3. (15pts) By definition, the derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ at $\mathbf{x}_0 \in \mathbb{R}^n$ is the unique linear function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0.$$

Of course, we call this function $\mathbf{T} = \mathbf{D}f(\mathbf{x}_0)$. I asserted in lecture that $\mathbf{T}$ is given by matrix multiplication by the matrix of partial derivatives at $\mathbf{x}_0$:

$$\mathbf{T}\,\mathbf{v} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Prove this.

(Hint: Show that T_{ij} satisfies

$$\lim_{h\to 0} \frac{|f_i(x_1,\dots,x_j+h,\dots,x_n) - f_i(x_1,\dots,x_n) - T_{ij}h|}{|h|} = 0$$

which is equivalent to the statement

$$T_{ij} = \lim_{h\to 0} \frac{f_i(x_1,\dots,x_j+h,\dots,x_n) - f_i(x_1,\dots,x_j,\dots,x_n)}{h} = \frac{\partial f_i}{\partial x_j}(x_1,\dots,x_n)$$

where $\mathbf{x}_0 = (x_1,\dots,x_n)$.)

Proof. We can take $\mathbf{x} \to \mathbf{x}_0$ however we like, so fix j and let

$$\mathbf{x} = \mathbf{x}_0 + h\hat{\mathbf{e}}_j$$

Then

$$\mathbf{x} - \mathbf{x}_0 = (0,\dots,0,\underbrace{h}_{j\text{th}},0,\dots,0) = h\hat{\mathbf{e}}_j$$

and

$$\|\mathbf{x} - \mathbf{x}_0\| = \sqrt{(h)^2} = |h|$$

Also, we have

$$\mathbf{T}(\mathbf{x} - \mathbf{x}_0) = h\,\mathbf{T}\,\hat{\mathbf{e}}_j = h\begin{bmatrix} T_{1j} \\ \vdots \\ T_{mj} \end{bmatrix}.$$

Now fix i, and note that

$$\begin{aligned}
&\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\| = \\
&\big((f_1(\mathbf{x}) - f_1(\mathbf{x}_0) - T_{1j}h)^2 + \cdots + (f_i(\mathbf{x}) - f_i(\mathbf{x}_0) - T_{ij}h)^2 + \cdots + (f_m(\mathbf{x}) - f_m(\mathbf{x}_0) - T_{mj}h)^2\big)^{1/2} \\
&\geq |f_i(\mathbf{x}) - f_i(\mathbf{x}_0) - T_{ij}h|
\end{aligned}$$

since all other terms are nonnegative. Thus

$$\lim_{h\to 0} \frac{|f_i(x_1,\dots,x_j+h,\dots,x_n) - f_i(x_1,\dots,x_n) - T_{ij}h|}{|h|} \leq \lim_{h\to 0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

But this is equivalent to saying that T_{ij} satisfies the definition of the partial derivative (you didn't have to show this part), since

$$\begin{aligned}
&\lim_{h\to 0} \frac{|f_i(x_1,\dots,x_j+h,\dots,x_n) - f_i(x_1,\dots,x_n) - T_{ij}h|}{|h|} = 0 \\
&\iff \lim_{h\to 0} \frac{f_i(x_1,\dots,x_j+h,\dots,x_n) - f_i(x_1,\dots,x_n) - T_{ij}h}{h} = 0 \\
&\iff \lim_{h\to 0} \frac{T_{ij}h}{h} = T_{ij} = \lim_{h\to 0} \frac{f_i(x_1,\dots,x_j+h,\dots,x_n) - f_i(x_1,\dots,x_n)}{h} = \frac{\partial f_i}{\partial x_j}(x_1,\dots,x_n)
\end{aligned}$$

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