Math 350 Problem Set 5 (due Friday 10/8 by 3pm)

Part I

1. (10pts) Show that
$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \, dy \right) dx = \frac{\pi}{4}$$
while
$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \, dx \right) dy = -\frac{\pi}{4}.$$
You may use the fact that
$$\int_0^1 \frac{s^2 - t^2}{(s^2+t^2)^2} \, dt = \frac{1}{1+s^2}.$$
(You may optionally try and show this yourself: first write $-t^2$ in the numerator as $t^2 - 2t^2$, split the integral into one of $\frac{1}{s^2+t^2}$, and one of $\frac{-2t^2}{(s^2+t^2)^2}$. Integrate the second integral by parts, giving the boundary term $\frac{1}{1+s^2}$, and a new integral which cancels the first integral.)
Why doesn't this violate either version of Fubini's theorem (Theorem 3 or 3')?

2. (10pts) Let $A \subset \mathbb{R}^2$. Suppose $f(x, y)$ is continuous and non-negative: $f(x, y) \geq 0$. Prove that if $\iint_A f(x, y) \, dA = 0$, then $f(x, y) = 0$ for all $(x, y) \in A$.
(Hint: give a proof by contradition. That is, assume $f(x_0, y_0) > 0$ for some (x_0, y_0), and then show that $\iint_A f(x, y) \, dA$ cannot equal 0. You may also want to use the monotonicity and additivity properties of the integral. Also note that you will have to use the assumption of continuity in an essential way: think of the function which is equal to 0 everywhere except a single point; this function has integral 0 (why?), but does not vanish everywhere.)

3. (10pts) Let $R = [0, 1] \times [0, 1]$ and let $f : R \to \mathbb{R}$ be the function
$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are rational numbers,} \\ 0 & \text{otherwise.} \end{cases}$$
Show that f is *not* integrable, by showing that the sequence of Riemann sums does not tend to a unique limit which is independent of the choice of points $\mathbf{c}_{jk}$.

Part II

1. (5pts) Evaluate
$$\iint_R x^m \, y^n \, dA, \quad m, n > 0$$
where $R = [0, 1] \times [0, 1]$.

2. (5pts) Compute the volume of the solid bounded by the xz plane, the yz plane, the xy plane, the planes $x = 1$ and $y = 1$, and the surface $z = x^2 + y^4$.

3. (5pts) Evaluate
$$\int_{-3}^{2} \int_0^{y^2} \left(x^2 + y \right) \, dx \, dy$$
and sketch the corresponding region.

4. (5pts) Let D be the region bounded by the positive x and y axes and the line $3x + 4y = 10$. Compute
$$\iint_D \left(x^2 + y^2 \right) \, dA.$$