Math 350 Problem Set 5 Solutions

Part I

1. (10pts) Show that

\[ \int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) \, dx = \frac{\pi}{4} \]

while

\[ \int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \right) \, dy = -\frac{\pi}{4}. \]

You may use the fact that

\[ \int_0^1 \frac{s^2 - t^2}{(s^2 + t^2)^2} \, dt = \frac{1}{1 + s^2}. \]

Why doesn’t this violate either version of Fubini’s theorem (Theorem 3 or 3’)?

\[ \text{Proof. The identity above is obtained as follows (you did not need to show this):} \]

\[ \int_0^1 \frac{s^2 - t^2}{(s^2 + t^2)^2} \, dt = \int_0^1 \frac{s^2 + t^2 - 2t^2}{(s^2 + t^2)^2} \, dt \]

\[ = \int_0^1 \frac{1}{s^2 + t^2} \, dt + \int_0^1 t \cdot \frac{-2t}{(s^2 + t^2)^2} \, dt \]

\[ = \int_0^1 \frac{1}{s^2 + t^2} \, dt + \frac{t}{s^2 + t^2} \bigg|_0^1 - \int_0^1 \frac{1}{s^2 + t^2} \, dt \]

\[ = \frac{1}{s^2 + 1}. \]

using integration by parts, where

\[ \frac{-2t}{(s^2 + t^2)^2} = \frac{d}{dt} \left( \frac{1}{s^2 + t^2} \right). \]

Interchanging \( s \) and \( t \), we see that

\[ \int_0^1 \frac{s^2 - t^2}{(s^2 + t^2)^2} \, ds = -\int_0^1 \frac{t^2 - s^2}{(s^2 + t^2)^2} \, ds = -\frac{1}{1 + t^2}. \]

Thus,

\[ \int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) \, dx = \int_0^1 \frac{1}{1 + x^2} \, dx = \arctan x \bigg|_{x=0}^{x=1} = \frac{\pi}{4} \]

whereas

\[ \int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \right) \, dy = \int_0^1 \frac{1}{1 + y^2} \, dy = -\arctan y \bigg|_{y=0}^{y=1} = -\frac{\pi}{4}. \]

This does not violate either version of Fubini’s theorem since the integrand is neither continuous nor bounded near \((0,0)\).
2. (10pts) Let $A \subset \mathbb{R}^2$. Suppose $f(x, y)$ is continuous and non-negative: $f(x, y) \geq 0$. Prove that if $\iint_A f(x, y) \, dA = 0$, then $f(x, y) = 0$ for all $(x, y) \in A$.

Proof. Assume $f(x_0, y_0) = c > 0$. We will show that $\iint_A f \, dA \neq 0$ (which is the contrapositive of the statement we’re trying to prove, hence equivalent). By continuity of $f$, there exists a $\delta > 0$ such that

$$\|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - f(x_0, y_0)| < \frac{c}{2} \implies f(x, y) > \frac{c}{2}.$$ 

Since

$$|x - x_0| < \delta / \sqrt{2} \text{ and } |y - y_0| < \delta / \sqrt{2} \implies \|(x, y) - (x_0, y_0)\| < \delta,$$

the disk $\|(x, y) - (x_0, y_0)\| < \delta$ contains a rectangle $R$ of area $2\delta^2 = \left(\frac{2\delta}{\sqrt{2}}\right) \left(\frac{2\delta}{\sqrt{2}}\right)$. Let $B = A \setminus R$ be the region obtained by deleting $R$ from $A$. By additivity and monotonicity ($f \geq 0$ and $f \geq \frac{c}{2}$ on $R$),

$$\iint_A f \, dA = \iint_B f \, dA + \iint_R f \, dA \geq 0 + (2\delta) \frac{c}{2} > 0.$$

and therefore $\iint_A f \, dA \neq 0$.

Assuming we know additivity for more general regions, we could alternatively just let $R$ be the disk $\|(x, y) - (x_0, y_0)\| < \delta$. \hfill \Box

3. (10pts) Let $R = [0, 1] \times [0, 1]$ and let $f : R \to \mathbb{R}$ be the function

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are rational numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f$ is not integrable, by showing that the sequence of Riemann sums does not tend to a unique limit which is independent of the choice of points $c_{jk}$.

We will produce two convergent sequences of Riemann approximations, which converge to different values. Let

$$S_n = \sum_{i,j=1}^n f(c_{ij}) \Delta x \Delta y$$

where $c_{ij} = (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and $x$ and $y$ are rational. On the other hand, let

$$S'_n = \sum_{i,j=1}^n f(c'_{ij}) \Delta x \Delta y$$

where $c'_{ij} = (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and at least one of $x$ or $y$ is irrational. This is possible since there are both rational and irrational points in any interval $[a, b]$ as long as $b > a$.

Thus

$$S_n = \sum_{i,j=1}^n 1 \Delta x \Delta y = \sum_{i,j=1}^n \frac{1}{n^2} = \frac{n^2}{n^2} = 1$$

and

$$S'_n = \sum_{i,j=1}^n 0 \Delta x \Delta y = 0.$$
Doing this for every $n$, we obtain sequences

$$\{S_n\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty} \quad \text{and} \quad \{S'_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$$

both of which converge, but

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} 1 = 1 \neq \lim_{n \to \infty} S'_n = \lim_{n \to \infty} 0 = 0.$$  

Thus $f$ is not integrable.