Math 350 Problem Set 6 (due Friday 10/15 by 3pm)

Part I

1. **Additivity over general regions**. Call a region $A \subset \mathbb{R}^2$ "nice" if its boundary consists of a piecewise set of differentiable curves. Let A and B be nice regions, with $B \subset A$. Let C be the set of points which are in A but not in B:
$$C = A \setminus B = \{(x,y) \mid (x,y) \in A, (x,y) \notin B\}$$
   (a) (8pts) Show that, if f is continuous (or even bounded, with a nice set of discontinuities), then
$$\iint_A f\,dA = \iint_B f\,dA + \iint_C f\,dA.$$
   (Hint: Recall how $\iint_A f\,dA$ is defined: $\iint_A f\,dA = \iint_R f^*\,dA$ for an auxilliary function f^* such that $f^* = f$ on A and 0 otherwise. If f^* is such a function for A, and f^{**} is such a function for B, what is $f^* - f^{**}$?)

   (b) (7pts) Iterate this to prove that, if $A = \bigcup_{i=1}^N A_i$ is a decomposition of a nice set A into N pieces which are also nice, then
$$\iint_A f\,dA = \sum_{i=1}^N \iint_{A_i} f\,dA.$$

2. **Generalized cones**. Let R be a nice region of $\mathbb{R}^2$. The cone over R, with height h, is the set of all points between $(0,0,h)$ and $(x,y,0)$ where $(x,y) \in R$:
$$C_h(R) = \{t(0,0,h) + (1-t)(x,y,0) \mid (x,y) \in R, 0 \le t \le 1\}.$$
   (a) (3pts) Draw a sketch of $C_h(R)$, so you can see what's going on.

   (b) (12pts) By setting up and evaluating a triple integral, prove that
$$\mathrm{Vol}(C_h(R)) = \frac{1}{3} h \mathrm{Area}(R)$$
   (Hints: Think about the cross sections $z =$ const. How do lengths in these cross sections scale with z? How does the area of the cross section scale with z? Your integral will probably be an iterated integral in dz and dA.)

Part II

1. (6pts) Sketch, and then compute the area of the region
$$R = \left\{(x,y) \,\middle|\, -\frac{1}{2}(y^2+1) \le x \le \frac{1}{2}(y^2+1),\ -\frac{1}{2}(x^2+1) \le y \le \frac{1}{2}(x^2+1)\right\}.$$

2. (6pts) Let D be a hemispherical region with radius 1:
$$D = \{(x,y,z) \mid z \ge 0,\ x^2+y^2+z^2 \le 1\}$$
   Suppose D has mass density given by $\delta(x,y,z) = z$. What is the total mass of D?

   You may end up doing a trig. substitution and needing to integrate $\sin^4(\theta)$. The following identity may be useful:
$$\sin^4(\theta) = \frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8}.$$

3. (6pts) Let D be the spatial region defined by $0 \le x \le 1 - y^2$, $-1 \le y \le 1$, and $0 \le z \le 1$. Find the center of mass of D, which is the point $(\overline{x}, \overline{y}, \overline{z})$, where
$$\overline{x} = \frac{1}{\mathrm{Vol}(D)} \iiint_D x\,dV, \quad \overline{y} = \frac{1}{\mathrm{Vol}(D)} \iiint_D y\,dV, \quad \overline{z} = \frac{1}{\mathrm{Vol}(D)} \iiint_D z\,dV.$$