Math 350 Problem Set 7 (due Friday 10/22 by 3pm)

Part I

1. (10pts) **Scaling property of multiple integrals**. Let $D \subset \mathbb{R}^3$ be a nice 3-dimensional region, with volume
$$\text{Vol}(D) = \iiint_D dV.$$
Let D_a be the region obtained by scaling (multiplying) all lengths in D by a factor $a \geq 0$:
$$D_a = \{a\mathbf{x} \mid \mathbf{x} \in D\}.$$
Use the change of variables theorem to prove that
$$\text{Vol}(D_a) = a^3\text{Vol}(D).$$
(Though we haven't discussed the general change of variables theorem for n-dimensional integrals except for $n \in \{1, 2, 3\}$, the same argument shows that if $D \in \mathbb{R}^n$, then the n-volume of D_a is a^n times the n-volume of D.)

2. (10pts) **4-volume of a 4-ball**. The 4-volume of a region $D \in \mathbb{R}^4$ is given by the integral
$$\text{Vol}_4(D) = \iiiint_D 1\, dV = \iiiint_D 1\, dx\, dy\, dz\, dw$$
The *4-sphere of radius a* is the set of points of distance a from the origin in $\mathbb{R}^4$:
$$S_a^4 = \left\{\mathbf{v} = (x, y, z, w) \in \mathbb{R}^4 \mid \|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2 + w^2} = a\right\}.$$
It's interior is called the *4-ball* (of radius a), and consists of the set
$$B_a^4 = \left\{\mathbf{v} = (x, y, z, w) \in \mathbb{R}^4 \mid \|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2 + w^2} \leq a\right\}.$$
Compute $\text{Vol}_4(B_a^4)$ by setting up an iterated integral.
(Hint: Use w for your inner integral, and note that the projection (shadow region) of B_a^4 onto the (x, y, z) space is equal B_a^3, the interior of the usual sphere of radius a. Spherical coordinates therefore might be useful for the rest of the integral. Also possibly useful will be the trig identity
$$\cos^2 t \sin^2 t = \frac{1 - \cos 4t}{8}.$$

3. (15pts) n**-volume of an** n**-ball**. Continuing in the above manner gets hard quickly, at least without some kind of appropriate spherical coordinates for all n. However, there is a neat trick to obtain a formula for $\text{Vol}_n(B_a^n)$ for any n. Here are some steps:
   (a) Argue (using an n dimensional analogue of problem 1, for instance), that
   $$\text{Vol}_n(B_a^n) = C_n a^n$$
   for some constant C_n, which is therefore all we need to find.
   (b) Write down an equation which computes $\text{Vol}_n(B_a^n)$ as a single integral, where the integrand consists of the $(n-1)$-volumes of $(n-1)$-balls of appropriate radii. Show that this gives a recursive formula for C_n in terms of C_{n-1}, but the integral is quite difficult to evaluate in general; you need not evaluate it.

(c) Do the recursion one more time, giving C_n in terms of C_{n-2} and an appropriate double integral. Note that *this* integral *is* easy to evaluate (hint: polar coordinates!). Evaluate it.

(d) Using values of C_n for small n that you know, write down the formula for $\mathrm{Vol}_n(B_a^n)$ for n up to $n = 10$. Impress your friends with this list.

(Note: it is similarly straightforward to find a 2-step recursive formula for the n-area of the n-sphere S_a^n. You might also do this for fun, but it is not required.)

Part II

1. (5pts) Let D be a hemispherical region with radius 1:

$$D = \{(x, y, z) \mid z \geq 0,\ x^2 + y^2 + z^2 \leq 1\}$$

Suppose D has mass density given by $\delta(x, y, z) = z$. Compute the total mass of D. Yes, this is the same problem as on PS6. Use whichever coordinates you did not use last time (cartesian or spherical).

2. (5pts) Let $f(x, y) = 2y^2$, and let R be the region bounded by the curves $y = x$, $y = 2x$, $y = 1/x$ and $y = 2/x$. Sketch R. Compute

$$\iint_R f\, dA$$

using a change of variables which results in an integral over a rectangular region.

3. (5pts) Let D be a cone, with its tip at $(0, 0, 0)$ and circular base of radius h lying in the plane $z = h$. (This is a *right* circular cone, meaning that the cone angle is $\pi/2$.) Compute

$$\iiint_D 3z^2\, dV.$$

4. (5pts) Show that the rotational moment of inertia of a uniform (meaning constant density) ball of radius a about a central axis is

$$I = \frac{2}{5} M a^2$$

where M is the mass. Recall that the moment of inertia of a body D about the z-axis is given by

$$I_z = \iiint_D (x^2 + y^2)\delta(x, y, z)\, dV$$

where δ is the density. In case it is useful, you may use

$$\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t.$$