Math 350 Problem Set 7 Solutions

Part I

1. (10pts) **Scaling property of multiple integrals**. Let $D \subset \mathbb{R}^3$ be a nice 3-dimensional region, with volume

$$\mathrm{Vol}(D) = \iiint_D dV.$$

Let D_a be the region obtained by scaling (multiplying) all lengths in D by a factor $a \geq 0$:

$$D_a = \{a\mathbf{x} \mid \mathbf{x} \in D\}.$$

Use the change of variables theorem to prove that

$$\mathrm{Vol}(D_a) = a^3 \mathrm{Vol}(D).$$

(Though we haven't discussed the general change of variables theorem for n-dimensional integrals except for $n \in \{1, 2, 3\}$, the same argument shows that if $D \in \mathbb{R}^n$, then the n-volume of D_a is a^n times the n-volume of D.)

Solution. Here's the proof for general n. Let $\mathbf{u}(u_1, \ldots, u_n) = a\mathbf{x} = (ax_1, \ldots, ax_n)$, so $\mathbf{u} \in D_a \iff \mathbf{x} \in D$ by definition of D_a. Then

$$\left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right| = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} = \begin{vmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{vmatrix} = |a^n| = a^n.$$

So

$$\mathrm{Vol}(D_a) = \int \cdots \int_{D_a} d\mathbf{u} = \int \cdots \int_D \left|\frac{\partial(\mathbf{u})}{\partial(\mathbf{x})}\right| d\mathbf{x} = a^n \int \cdots \int_D d\mathbf{x} = a^n \mathrm{Vol}(D).$$

2. (10pts) **4-volume of a 4-ball**. The 4-volume of a region $D \in \mathbb{R}^4$ is given by the integral

$$\mathrm{Vol}_4(D) = \iiiint_D 1\, dV = \iiiint_D 1\, dx\, dy\, dz\, dw$$

The *3-sphere of radius* a is the set of points of distance a from the origin in $\mathbb{R}^4$:

$$S_a^3 = \left\{\mathbf{v} = (x, y, z, w) \in \mathbb{R}^4 \mid \|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2 + w^2} = a\right\}.$$

It's interior is called the *4-ball* (of radius a), and consists of the set

$$B_a^4 = \left\{\mathbf{v} = (x, y, z, w) \in \mathbb{R}^4 \mid \|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2 + w^2} \leq a\right\}.$$

Compute $\mathrm{Vol}_4(B_a^4)$ by setting up an iterated integral.

(Hint: Use w for your inner integral, and note that the projection (shadow region) of B_a^4 onto the (x, y, z) space is equal B_a^3, the interior of the usual sphere of radius a. Spherical coordinates therefore might be useful for the rest of the integral. Also possibly useful will be the trig identity

$$\cos^2 t \sin^2 t = \frac{1 - \cos 4t}{8}.$$

Solution. We can compute $\mathrm{Vol}_4(B_a^4)$ by the integral

$$
\begin{aligned}
\iiiint_{B_a^4} dV_4 &= \iiint_{B_a^3} \int_{-\sqrt{a^2-x^2-y^2-z^2}}^{\sqrt{a^2-x^2-y^2-z^2}} dw\, dV \\
&= 2\iiint_{B_a^3} \int_0^{\sqrt{a^2-\rho^2}} dw\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= 2\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\int_0^{\sqrt{a^2-\rho^2}} dw\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= 4\pi \int_0^{\pi} \sin\phi\, d\phi \int_0^a \rho^2\sqrt{a^2-\rho^2}\, d\rho \\
&= 8\pi \int_0^{\pi/2} a^2\sin^2 t\sqrt{a^2-a^2\sin^2 t}\, d(a\sin t) \\
&= 8\pi \int_0^{\pi/2} a^4 \sin^2 t\cos^2 t\, dt \\
&= \frac{\pi^2}{2}a^4.
\end{aligned}
$$

3. (15pts) n-**volume of an** n-**ball**. Continuing in the above manner gets hard quickly, at least without some kind of appropriate spherical coordinates for all n. However, there is a neat trick to obtain a formula for $\mathrm{Vol}_n(B_a^n)$ for any n. Here are some steps:

   (a) Argue (using an n dimensional analogue of problem 1, for instance), that

   $$\mathrm{Vol}_n(B_a^n) = C_n a^n$$

   for some constant C_n, which is therefore all we need to find.

   (b) Write down an equation which computes $\mathrm{Vol}_n(B_a^n)$ as a single integral, where the integrand consists of the $(n-1)$-volumes of $(n-1)$-balls of appropriate radii. Show that this gives a recursive formula for C_n in terms of C_{n-1}, but the integral is quite difficult to evaluate in general; you need not evaluate it.

   (c) Do the recursion one more time, giving C_n in terms of C_{n-2} and an appropriate double integral. Note that *this* integral *is* easy to evaluate (hint: polar coordinates!). Evaluate it.

   (d) Using values of C_n for small n that you know, write down the formula for $\mathrm{Vol}_n(B_a^n)$ for n up to $n = 10$. Impress your friends with this list.

   (Note: it is similarly straightforward to find a 2-step recursive formula for the n-volume of the n-spheres S_a^n. You might also do this for fun, but it is not required.)

   Solution. Let
   $$V_n(a) = \mathrm{Vol}_n(B_a^n).$$
   It follows from problem 1 that
   $$V_n(a) = C_n a^n \tag{1}$$
   for some constant C_n which depends only on n. We can write $V_n(a)$ as an integral over one of the cartesian variables, say x_n, obtaining
   $$V_n(a) = \int_{-a}^{a} V_{n-1}\left(\sqrt{a^2-x_n^2}\right) dx_n, \tag{2}$$

since the intersection of B_a^n with the plane $x_n = c$ is the $(n-1)$ ball $\left\{x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \le a^2 - c^2\right\}$. Plugging in (1) into (2), we obtain

$$C_n a^n = \int_{-a}^{a} C_{n-1} \left(a^2 - x_n^2\right)^{(n-1)/2} \, dx_n,$$

which computes C_n in terms of C_{n-1} in principle, but the integral is difficult to compute. However, using the recursion one additional time (that is to say, plugging (2) into itself), we get

$$\begin{aligned} V_n(a) = C_n a^n &= \int_{-a}^{a} \int_{-\sqrt{a^2-x_n^2}}^{\sqrt{a^2-x_n^2}} V_{n-2}\left(\sqrt{\left(\sqrt{a^2 - x_n^2}\right)^2 - x_{n-1}^2}\right) dx_{n-1}\, dx_n \\ &= \int_{-a}^{a} \int_{-\sqrt{a^2-x_n^2}}^{\sqrt{a^2-x_n^2}} V_{n-2}\left(\sqrt{a^2 - x_{n-1}^2 - x_n^2}\right) dx_{n-1}\, dx_n \\ &= \int_{-a}^{a} \int_{-\sqrt{a^2-x_n^2}}^{\sqrt{a^2-x_n^2}} C_{n-2} \left(a^2 - x_{n-1}^2 - x_n^2\right)^{(n-2)/2} dx_{n-1}\, dx_n. \end{aligned}$$

This is just an integral over a disk of radius a in the x_{n-1}, x_n plane, which we can evaluate using polar coordinates (r, θ) in this plane:

$$C_n a^n = \int_0^{2\pi} \int_0^a C_{n-2} \left(a^2 - r^2\right)^{(n-2)/2} r\, dr\, d\theta = 2\pi C_{n-2} \left[-\frac{2}{n}\frac{1}{2}\left(a^2 - r^2\right)^{n/2}\right]_{r=0}^{a} = \frac{2\pi}{n} C_{n-2} a^n.$$

We cancel the a^n to get

$$C_n = \frac{2\pi}{n} C_{n-2}.$$

Using $C_0 = 1$ and $C_1 = 2$, we obtain

n	C_n	$V_n(a)$
0	1	1
1	2	$2a$
2	π	πa^2
3	$\frac{4\pi}{3}$	$\frac{4\pi}{3} a^3$
4	$\frac{\pi^2}{2}$	$\frac{\pi^2}{2} a^4$
5	$\frac{8\pi^2}{15}$	$\frac{8\pi^2}{15} a^5$
6	$\frac{\pi^3}{6}$	$\frac{\pi^3}{6} a^6$
7	$\frac{16\pi^3}{105}$	$\frac{16\pi^3}{105} a^7$
8	$\frac{\pi^4}{24}$	$\frac{\pi^4}{24} a^8$
9	$\frac{32\pi^4}{945}$	$\frac{32\pi^4}{945} a^9$
10	$\frac{\pi^5}{120}$	$\frac{\pi^5}{120} a^{10}$
$2n$	$\frac{\pi^n}{n!}$	$\frac{\pi^n}{n!} a^{2n}$
$2n+1$	$\frac{2^{n+1}\pi^n}{(2n+1)!!}$	$\frac{2^{n+1}\pi^n}{(2n+1)!!} a^{2n+1}$

emplying the *double factorial*, $(2n+1)!! = (2n+1)(2n-1)\cdots 5 \cdot 3 \cdot 1$.