Math 350 Problem Set 8 (due Friday 11/5 by 3pm)

Part I

1. Find the values of $a \geq 0$ such that the following integrals exist. Justify your answers

   (a) (6pts) $D_1 = \{(x, y) \mid x^2 + y^2 \leq 1\}$

   $$I_a = \iint_{D_1} \frac{1}{(x^2 + y^2)^a}\, dA$$

   (b) (6pts) $D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ y \leq x\}$,

   $$I_a = \iint_{D_2} \frac{1}{(x - y)^a}\, dA$$

2. (6pts) Show that if $\mathcal{C}$ is a curve defined in polar coordinates by $r = r(\theta), \theta_0 \leq \theta \leq \theta_1$, then the path integral of $f(x, y)$ over $\mathcal{C}$ is given by

   $$\int_{\mathcal{C}} f\, ds = \int_{\theta_0}^{\theta_1} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + \frac{dr}{d\theta}^2}\, d\theta.$$

   (Hint: how is this curve parametrized? Don't let yourself get confused just because the parametrizing variable is a different one than you're used to!)

3. (6pts) Consider the spherical surface $\rho = a$ where $a \in \mathbb{R}$ is constant. Show that, in terms of variables (ϕ, θ),

   $$dS = a^2 \sin\phi\, d\phi\, d\theta$$

   on this surface.

4. (6pts) Consider the surface $\phi = \alpha$ where $\alpha \in [0, \pi]$ is constant. What does this surface look like? Show that, in terms of variables (ρ, θ),

   $$dS = \rho \sin\alpha\, d\rho\, d\theta$$

   (Note that there is only a single power of ρ!).

5. (10pts) On a surface defined by $z = g(x, y)$, our formula for the oriented surface area element is

   $$\mathbf{n}\, dS = d\mathbf{S} = (-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k})\, dx\, dy,$$

   where $g_x = \frac{\partial g}{\partial x}(x, y)$ and $g_y = \frac{\partial g}{\partial y}(x, y)$. Use this to show that if the same surface is also defined by the equation $f(x, y, z) = c$ for some constant c, then

   $$\mathbf{n}\, dS = d\mathbf{S} = \frac{\nabla f}{f_z}\, dx\, dy.$$

Part II

1. Let $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, and $\mathcal{C}$ be the oriented curve defined by $(\cos t, \sin t, t)$ for $0 \leq t \leq 2\pi$. Let

   $$I = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}.$$

   (a) (3pts) Evaluate I directly.

(b) (3pts) $\mathbf{F}(x, y, z)$ is in fact a conservative vector field. Evaluate I by finding a potential function f such that $\mathbf{F} = \nabla f$ and use the fundamental theorem of calculus for line integrals.

2. (6pts) Compute the path integral

$$\int_{\mathbf{c}} \frac{(x+y)}{(y+z)}\, ds, \quad \mathbf{c}(t) = (t, 2/3t^{3/2}, t),\ 1 \leq t \leq 2$$

3. (6pts) Compute the line integral

$$\int_{\mathbf{c}} x\, dy - y\, dx, \quad \mathbf{c}(t) = (\cos t, \sin t),\ 0 \leq t \leq \pi/2$$

(You may want to write it in the form $\mathbf{F} \cdot d\mathbf{s}$ first if that helps you.)

4. Find the surface area of the following surfaces:

   (a) (4pts) $z = 4 - x^2 - y^2$, $z \geq 0$.

   (b) (4pts) The sphere of radius a (Hint: you may want to use the result from problem 3 in Part I)

   (c) (4pts) The cone $z = \sqrt{x^2 + y^2}$, $z \leq 1$. (Hint: problem 4 in Part I gives one possibility)

5. (4pts) Compute the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the oriented surface consisting of the upper unit hemisphere, with upward pointing normal.