Math 350 Problem Set 9 Solutions

Part I

1. (5pts) Prove that the curl of a gradient is always zero, and that the divergence of a curl is always zero (assuming f or $\mathbf{F}$ are C^2 functions):

$$\nabla \times (\nabla f) = 0, \quad \text{and} \quad \nabla \cdot (\nabla \times \mathbf{F}) = 0 \qquad \text{for all } f \text{ and } \mathbf{F}.$$

The divergence of a gradient is not necessarily zero, however. The **Laplacian** is defined to be the operator Δ which takes a scalar function to a scalar function by the formula

$$\Delta f = \nabla \cdot (\nabla f).$$

Write down the expression for Δf in 1, 2 and 3 dimensions.

Solution. For the curl of a gradient, we have

$$\nabla \times (\nabla f) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) \mathbf{k} = \mathbf{0}$$

by symmetry of mixed partial derivatives.

Similarly, if $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$, we have

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}$$

and so

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \\ &= 0 \end{aligned}$$

since partial derivatives are symmetric and each term appears twice, with opposite signs.

For the Laplacian, we compute

$$\Delta f = \nabla \cdot (\nabla f) = \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

In 1 and two dimensions we have

$$\Delta f = \frac{d^2}{dx^2} f(x), \quad \text{and} \quad \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

respectively.

2. (10pts) Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be a scalar function. Use Green's Theorem to prove the formula

$$\iint_R f \Delta f + \nabla f \cdot \nabla f \, dA = \oint_{\partial R} f \nabla f \cdot \hat{\mathbf{n}} \, ds$$

Solution. Define a vector field by

$$\mathbf{F}(x,y) = f(x,y)\nabla f(x,y) = f\frac{\partial f}{\partial x}\mathbf{i} + f\frac{\partial f}{\partial y}\mathbf{j}$$

We use the flux form of Green's Theorem, which says

$$\oint_{\partial R} f\nabla f\cdot\hat{\mathbf{n}}\,ds = \oint_{\partial R}\mathbf{F}\cdot\hat{\mathbf{n}}\,ds = \iint_R \nabla\cdot\mathbf{F}\,dA.$$

To compute $\nabla\cdot\mathbf{F}$, we have

$$\begin{aligned}
\nabla\cdot\mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right)\cdot\left(f(x,y)\frac{\partial f}{\partial x}(x,y)\mathbf{i} + f(x,y)\frac{\partial f}{\partial y}(x,y)\mathbf{j}\right)\\
&= \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} + f\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial y^2}\\
&= f\Delta f + \nabla f\cdot\nabla f,
\end{aligned}$$

from which the result follows.

3. (15pts) **Laplace's Equation** is the (partial differential) equation $\Delta u = 0$, for a scalar function $u(x,y)$. Functions u which satisfy Laplace's equation (called **harmonic functions**) are deeply important in mathematics and physics. A typical problem that arises is to find a function u such that $\Delta u = 0$ on the interior of a region $R \subset \mathbb{R}^2$, and such that u is equal to some fixed function at the boundary of R, i.e. that $u(x,y) = g(x,y)$ for all $(x,y)\in\partial R$. Such a u is said to be a solution of the **boundary value problem**

$$\begin{cases}\Delta u(x,y) = 0 & (x,y)\in R\\ u(x,y) = g(x,y) & (x,y)\in\partial R\end{cases}\tag{1}$$

and g is called the **boundary value.**

This is a physical model for the following situation. Take the curve in $\mathbb{R}^3$ given by the graph of g over ∂R; that is, the set $\{(x,y,g(x,y))\in\mathbb{R}^3 \mid (x,y)\in\partial R\}$. Picture this curve in space as a rigid wire, and then dip this wire into a solution of soap and water, or imagine attaching a rubber sheet to it. The soap film or rubber sheet, stretched out in the space inside the wire ring, defines a surface which is a solution to (1).

For mathematical purposes, it is nice to know two things about such differential equation problems: A) that solutions *exist* (i.e. given a g, that at least one function u exists which solves (1), and B) that solutions are *unique* (so that there is only one solution for each choice of g). These questions can be very difficult to answer for general partial differential equations – in fact, much ongoing mathematical research today is concerned with such questions.

For Laplace's equation (1) however, we can prove uniqueness using the formula from problem 2. (Existence is harder, and somewhat beyond the scope of this class.)

(a) Suppose g is fixed, and u_1, u_2 are two solutions to (1). Show that $u_1 - u_2$ is a solution to Laplace's Equation, but with a different boundary value. What is the boundary value of $u_1 - u_2$?

(b) Set $f = u_1 - u_2$. Use the formula from problem 2 to show that

$$\iint_R \nabla f\cdot\nabla f\,dA = \iint_R \|\nabla f\|^2\,dA = 0.$$

(You will need to use the fact that f is harmonic, as well as the particular boundary value of f.)

(c) Conclude that we must have

$$\nabla f = 0.$$

(Remember that we showed a while back that if $\iint_R h\,dA = 0$ for a nonnegative, continuous function h, then $h = 0$).

(d) Thus $f = u_1 - u_2$ must be a constant in R. Argue that this constant must be 0 because of the boundary value of f. Therefore,

$$u_1 = u_2$$

and we have proved that solutions to (1) are in fact unique.

Solution. Inside R, we have

$$\Delta(u_1(x,y) - u_2(x,y)) = \Delta u_1(x,y) - \Delta u_2(x,y) = 0 - 0 = 0 \qquad (x,y) \in R$$

since u_1 and u_2 are solutions. On ∂R, we have

$$u_1(x,y) - u_2(x,y) = g(x,y) - g(x,y) = 0 \qquad (x,y) \in \partial R.$$

Hence $u_1 - u_2$ solves Laplace's Equation with boundary value 0. From problem 2,

$$\iint_R f\Delta f + \nabla f \cdot \nabla f\,dA - \oint_{\partial R} f\nabla f \cdot \hat{\mathbf{n}}\,ds = 0,$$

but since $\Delta f = 0$ in R and $f = 0$ on ∂R, this equals

$$0 + \iint_R \nabla f \cdot \nabla f\,dA - 0 = 0.$$

Since $\|\nabla f\|^2 = \nabla f \cdot \nabla f$ is a nonnegative, continuous function, we must have

$$\|\nabla f\|^2 = 0 \implies \nabla f = \mathbf{0},$$

and therefore $f = u_1 - u_2$ is a constant function. Since $f = 0$ on ∂R, this constant must be 0. Thus

$$u_1 = u_2 \quad \text{in R.}$$

Part II

1. Let $\mathbf{F}(x,y) = y\mathbf{i} - x\mathbf{j}$, and suppose $\mathcal{C}$ is a circle of radius 1 with center at $(1,0)$, oriented clockwise. Compute

$$I = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\,ds$$

two ways:

(a) (5pts) Directly.

Solution. Note that $\mathcal{C}$ has the opposite orientation from the usual one. We can either parametrize by

$$(x(t), y(t)) = (\cos t + 1, -\sin t), \quad 0 \le t \le 2\pi$$

or by

$$(x(t), y(t)) = (\cos t + 1, \sin t) \quad 0 \le t \le 2\pi$$

as long as we introduce a minus sign to compensate for the orientation. Using the first parametrization, we have

$$\begin{aligned}
\oint_{\mathcal{C}} \mathbf{F} \cdot bvT\,ds &= \int_0^{2\pi} (-\sin t\mathbf{i} - (\cos t + 1)\mathbf{j}) \cdot (-\sin t\mathbf{i} - \cos t\mathbf{j})\,dt \\
&= \int_0^{2\pi} \sin^2 t + \cos^2 t + \cos t\,dt \\
&= \int_0^{2\pi} 1 + \cos t\,dt = 2\pi.
\end{aligned}$$

(b) (5pts) Using Green's Theorem.

Solution. Let R be the disk $\{(x-1)^2+y^2 \leq 1\}$. By the orientation convention, we have

$$\partial R = -\mathcal{C}$$

since the boundary of ∂R must be oriented so that R is on the left. Thus

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = -\iint_R \nabla \times \mathbf{F}\, dA.$$

We compute

$$\nabla \times \mathbf{F} = \frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) = -2.$$

Since this is a constant, we have

$$-\iint_R \nabla \times \mathbf{F}\, dA = 2\iint_R dA = 2\text{Area}(R) = 2\pi.$$

2. Let $\mathbf{F}(x,y) = xy^2\mathbf{i} + xy\mathbf{j}$, and suppose $\mathcal{C}$ consists of the straight line segements from $(0,0)$ to $(1,0)$ to $(0,1)$ and back to $(0,0)$, oriented counterclockwise. Compute

$$I = \oint_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$$

two ways:

(a) (5pts) Directly.

Solution. Note that $\mathbf{F}(x,y)$ vanishes on the line segments from $(0,0)$ to $(1,0)$ and $(0,1)$ to $(0,0)$. Thus

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \int_{\mathcal{C}_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$$

where $\mathcal{C}_2$ is the segment from $(1,0)$ to $(0,1)$. We can parametrize $\mathcal{C}_2$ by

$$(x,y) = (-t, 1+t), \quad -1 \leq t \leq 0$$

(or some other choice, as long as the orientation is taken into account). Along $\mathcal{C}_2$ then,

$$\hat{\mathbf{n}}\, ds = dy\mathbf{i} - dx\mathbf{j} = (\mathbf{i} + \mathbf{j})\, dt$$

so

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \int_{-1}^{0} \left(-t(1+t)^2\mathbf{i} - t(1+t)\mathbf{j}\right) \cdot (\mathbf{i}+\mathbf{j})\, dt = -\int_{-1}^{0} t^3 + 3t^2 + 2t\, dt = \frac{1}{4}.$$

(b) (5pts) Using Green's Theorem.

Solution. We use the flux form of Green's Theorem, so

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

where R is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. We compute

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left(xy^2\right) + \frac{\partial}{\partial y}(xy) = y^2 + x,$$

so

$$\iint_R \nabla \cdot F\, dA = \int_0^1 \int_0^{1-y} y^2 + x\, dx\, dy = \frac{1}{4}.$$

3. (10pts) Find the closed curve $\mathcal{C}$ in $\mathbb{R}^2$ which *maximizes* the line integral

$$\oint_{\mathcal{C}} (x^2 - 2)y\, dx - (y^2 - 2)x\, dy.$$

That is, find the curve over which this integral has the largest possible value.

(Hint: Green's Theorem.)

Note: as stated, this problem is ill-posed, which I have just realized in the course of typing the solutions. By taking *clockwise* oriented curves which enclose larger and larger areas, we can make the integral arbitrarily large and positive. To fix this, we should ask for the *counterclockwise oriented* closed curve which maximizes the above integral.

Solution. Maximizing over all possible curves is hard. The problem becomes easier when we use Green's Theorem. For an arbitrary (counterclockwise oriented) closed curve $\mathcal{C}$

$$\oint_{\mathcal{C}} (x^2 - 2)y\, dx - (y^2 - 2)x\, dy = \iint_R \big((2 - y^2) - (x^2 - 2)\big)\, dA = \iint_R 4 - x^2 - y^2\, dA$$

where R is a region such that $\partial R = \mathcal{C}$.

The problem is now to find the region R such that the integral

$$\iint_R 4 - x^2 - y^2\, dA$$

is as large as possible. This happens when R consists of the set where the integrand is non-negative, namely $R = \{x^2 + y^2 \leq 4\}$, the circle of radius 2 centered at the origin. Thus the answer to the original question is

$$\mathcal{C} = \partial R = \text{circle of radius 2 centered at } (0,0), \text{ oriented CCW.}$$