MATH 540 EXAM 2 PRACTICE PROBLEMS

1. Theoretical exercises

Problem 1. Prove that if A is normal, then so are A^{T}, A^n for any n. Are the converses true? Can you find matrices such that A^T is normal but A is not, or such that A^2 is normal but A is not?

Problem 2. Prove that if T is unitary and self-adjoint, then the only eigenvalues of T can be ± 1. Do not assume the spectral theorem. Use the defining properties of unitarity and self-adjointness, along with properties of the determinant.

Problem 3. Suppose P is self-adjoint, and all of its eigenvalues are either 0 or 1. Show P is a projection, which is the identity if 0 is not an eigenvalue.

Problem 4. Let P be orthogonal projection onto a subspace E, and let Q be the projection onto $E^{\perp}$. Simplify the operator

$$9P^{83} + 3P^{12} + 13Q^{12} - Q$$

Problem 5. A linear transformation $A : V \longrightarrow W$ is called *injective* if no two vectors can go to the same point; i.e. $Ax_1 = Ax_2 \iff x_1 = x_2$. A is called *surjective* if every vector in W is in the image of A: i.e. $\mathrm{Ran} A = W$. Prove that

$$A \text{ is injective} \iff A^* \text{ is surjective.}$$

Problem 6. Use the trace to show that there cannot exist transformations T, S such that $TS - ST = I$. (Hint: think about eigenvalues).

Problem 7. Prove the following statements or find a counterexample:

(1) If $\mathbf{v}$ is an eigenvector of A then $\mathbf{v}$ is an eigenvector of A^n for any n.
(2) If $\mathbf{v}$ is an eigenvector of A, then $\overline{\mathbf{v}}$ is an eigenvector of A^*.
(3) If $\mathbf{v}$ is an eigenvector of A, then $\overline{\mathbf{v}}$ is an eigenvector of $\overline{A}$.

Problem 8. Suppose $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of an operator A. Let $\alpha \in \mathbb{C}$. What are the eigenvalues of $A + \alpha I$?

Problem 9. Let $M_{2\times 2}$ be the space of 2×2 real matrices. For any $A \in M_{2\times 2}$, let $T_A : M_{2\times 2} \longrightarrow M_{2\times 2}$ be the operator given by

$$T_A X = AX$$

(1) Let U be an invertible matrix. Show that if $X \in M_{2\times 2}$ is an eigenvector of T_A, then UX is an eigenvector of $T_{UAU^{-1}}$ with the same eigenvalue.
(2) Show that the eigenvalues of $T_A : M_{2\times 2} \longrightarrow M_{2\times 2}$ coincide with those of $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. What happens to the multiplicities? (Hints: show it directly, or use the first part of the problem to reduce to the case that A is upper triangular.)

2. Computational exercises

Problem 10. Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

That is, find matrices $P$and D such that P is orthogonal (so $P^{-1} = P^{\mathrm{T}}$), D is diagonal, and $A = PDP^{\mathrm{T}}$.

Problem 11. Find the best fit linear curve $y = ax+b$ for the data points $(x_n, y_n) \in \{(1,1), (-2,1), (3,4), (2,3)\}$.

Problem 12. Compute the following determinants:

(1)

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{vmatrix}$$

(2)

$$\begin{vmatrix} 2 & 4 & 9 & 6 \\ 0 & 2 & 4 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & -2 & -4 & 1 \end{vmatrix}$$

Problem 13. Diagonalize the matrix

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}$$

and find all possible square roots; i.e. complex 2×2 matrices B such that $B^2 = A$.

Problem 14. Compute the projections onto the four fundamental subspaces $\mathrm{Ran}A$, $\mathrm{Ker}A$, $\mathrm{Ran}A^*$, and $\mathrm{Ker}A^*$, where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 0 & 6 \end{pmatrix}$$

Problem 15. Which of the following pairs of matrices are similar? Which are unitarily equivalent?

(1) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$