Calc III: Workshop 3, Fall 2017

Problem 1. Let $\mathbf{f}(t)$ be a smooth curve such that $\mathbf{f}'(t) \neq \mathbf{0}$ for all t. The *unit tangent vector* to the curve is defined by

$$\mathbf{T}(t) = \frac{\mathbf{f}'(t)}{\|\mathbf{f}'(t)\|}.$$

The *unit normal vector* is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

and the *unit binormal* is defined by

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

At each time t, these form an orthogonal set of unit vectors along the curve, called the *Frenet frame.* (It is with respect to this frame for instance that spacecraft trajectory computations are carried out). Using the identities

$$\|\mathbf{g}(t)\|' = \frac{\mathbf{g}(t) \cdot \mathbf{g}'(t)}{\|\mathbf{g}(t)\|}, \quad \text{and} \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},$$

show that

$$\mathbf{T}'(t) = \frac{\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t))}{\|\mathbf{f}'(t)\|^3},$$

$$\mathbf{N}(t) = \frac{\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t))}{\|\mathbf{f}'(t)\| \, \|\mathbf{f}''(t) \times \mathbf{f}'(t)\|},$$

$$\mathbf{B}(t) = \frac{\mathbf{f}'(t) \times \mathbf{f}''(t)}{\|\mathbf{f}'(t) \times \mathbf{f}''(t)\|}.$$

Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ at each t for the helical curve $\mathbf{f}(t) = (\cos t, \sin t, t)$.

Problem 2.

(a) Calculate the arc length functions $s(t) = \int_a^t \|\mathbf{f}'(u)\| \, du$ for the curves $\mathbf{f}(t) = (3\cos 2t, 3\sin 2t, 3t)$, for $0 \leq t \leq \pi/2$, and $\mathbf{g}(t) = (2\cos 3t, 2\sin 3t, 2t^{3/2})$ for $0 \leq t \leq 1$.

(b) Find the arc length parameterizations $\mathbf{f}(s)$ and $\mathbf{g}(s)$.

Problem 3. If a curve $\mathbf{f}(s)$ is parameterized in terms of arc length, then $\frac{d}{ds}\|\mathbf{f}(s)\| = 1$, so the unit tangent becomes simply $\mathbf{T}(s) = \mathbf{f}'(s)$. The *curvature* of the curve is defined by

$$\kappa(s) = \left\| \frac{d\mathbf{T}(s)}{ds} \right\| = \left\| \frac{d^2\mathbf{f}(s)}{ds^2} \right\|.$$

Often, it is easier to compute using an arbitrary parameterization. Using the chain rule $\frac{d}{ds}\mathbf{T}(t(s)) = \frac{d}{dt}\mathbf{T}(t(s))\frac{dt}{ds}$ and the fact that $\frac{ds}{dt} = \|\mathbf{f}'(t)\|$, show that

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{f}'(t)\|}.$$

Show further that $\kappa(t)$ is given by

$$\kappa(t) = \frac{\|\mathbf{f}'(t) \times \mathbf{f}''(t)\|}{\|\mathbf{f}'(t)\|^3}.$$

Problem 4. Compute the curvature at each point for the helix $\mathbf{f}(t) = (\cos t, \sin t, t)$ and for the curves in Problem 2.