Calc III: Workshop 3 Solutions, Fall 2017

Problem 1. Let $\mathbf{f}(t)$ be a smooth curve such that $\mathbf{f}'(t) \neq \mathbf{0}$ for all t. The *unit tangent vector* to the curve is defined by

$$\mathbf{T}(t) = \frac{\mathbf{f}'(t)}{\|\mathbf{f}'(t)\|}.$$

The *unit normal vector* is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

and the *unit binormal* is defined by

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

At each time t, these form an orthogonal set of unit vectors along the curve, called the *Frenet frame*. (It is with respect to this frame for instance that spacecraft trajectory computations are carried out). Using the identities

$$\|\mathbf{g}(t)\|' = \frac{\mathbf{g}(t) \cdot \mathbf{g}'(t)}{\|\mathbf{g}(t)\|}, \quad \text{and} \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},$$

show that

$$\mathbf{T}'(t) = \frac{\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t))}{\|\mathbf{f}'(t)\|^3},$$

$$\mathbf{N}(t) = \frac{\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t))}{\|\mathbf{f}'(t)\| \, \|\mathbf{f}''(t) \times \mathbf{f}'(t)\|},$$

$$\mathbf{B}(t) = \frac{\mathbf{f}'(t) \times \mathbf{f}''(t)}{\|\mathbf{f}'(t) \times \mathbf{f}''(t)\|}.$$

Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ at each t for the helical curve $\mathbf{f}(t) = (\cos t, \sin t, t)$.

Solution. First we may compute that

$$\left(\frac{1}{\|\mathbf{g}(t)\|}\right)' = -\frac{\mathbf{g}(t) \cdot \mathbf{g}'(t)}{\|\mathbf{g}(t)\|^3},$$

and then, applying this to the case $\mathbf{g}(t) = \mathbf{f}'(t)$ and using the chain rule, we may compute

$$\begin{aligned} \mathbf{T}'(t) &= -\frac{(\mathbf{f}'(t) \cdot \mathbf{f}''(t))}{\|\mathbf{f}'(t)\|^3}\mathbf{f}'(t) + \frac{1}{\|\mathbf{f}'(t)\|}\mathbf{f}''(t) \\ &= -\frac{(\mathbf{f}'(t) \cdot \mathbf{f}''(t))}{\|\mathbf{f}'(t)\|^3}\mathbf{f}'(t) + \frac{(\mathbf{f}'(t) \cdot \mathbf{f}'(t))\mathbf{f}''(t)}{\|\mathbf{f}'(t)\|^3} \\ &= \frac{\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t))}{\|\mathbf{f}'(t)\|^3}. \end{aligned}$$

Next, using the fact that $\mathbf{f}''(t) \times \mathbf{f}'(t)$ is orthogonal to $\mathbf{f}'(t)$ in particular, we have $\|\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t))\| = \|\mathbf{f}'(t)\| \, \|\mathbf{f}''(t) \times \mathbf{f}'(t)\|$ since the sine of the angle is 1. Thus

$$\mathbf{N}(t) = \frac{\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t))}{\|\mathbf{f}'(t)\| \, \|\mathbf{f}''(t) \times \mathbf{f}'(t)\|}.$$

Finally, by definition of $\mathbf{B}(t)$ we have

$$\begin{aligned}\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) &= \frac{\mathbf{f}'(t) \times (\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t)))}{\|\mathbf{f}'(t)\|^2 \|\mathbf{f}''(t) \times \mathbf{f}'(t)\|} \\ &= -\frac{\|\mathbf{f}'(t)\|^2 (\mathbf{f}''(t) \times \mathbf{f}'(t))}{\|\mathbf{f}'(t)\|^2 \|\mathbf{f}''(t) \times \mathbf{f}'(t)\|} \\ &= -\frac{\mathbf{f}''(t) \times \mathbf{f}'(t)}{\|\mathbf{f}''(t) \times \mathbf{f}'(t)\|} = \frac{\mathbf{f}'(t) \times \mathbf{f}''(t)}{\|\mathbf{f}'(t) \times \mathbf{f}''(t)\|},\end{aligned}$$

where we have used the triple product formula $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ with $\mathbf{a} = \mathbf{b} = \mathbf{f}'(t)$ and $\mathbf{c} = (\mathbf{f}''(t) \times \mathbf{f}'(t))$, in which case $\mathbf{a} \cdot \mathbf{c} = 0$.
For the helix $\mathbf{f}(t) = (\cos t, \sin t, t)$, we obtain

$$\begin{aligned}\mathbf{f}'(t) &= (-\sin t, \cos t, 1) & \|\mathbf{f}'(t)\| &= \sqrt{2} \\ \mathbf{f}''(t) &= (-\cos t, -\sin t, 0) & \|\mathbf{f}''(t)\| &= 1 \\ \mathbf{f}''(t) \times \mathbf{f}'(t) &= (-\sin t, \cos t, -1) & \|\mathbf{f}''(t) \times \mathbf{f}'(t)\| &= \sqrt{2} \\ \mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t)) &= (-2\cos t, -2\sin t, 0)\end{aligned}$$

whence

$$\begin{aligned}\mathbf{T}(t) &= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1), \\ \mathbf{N}(t) &= (-\cos t, -\sin t, 0) \\ \mathbf{B}(t) &= \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1).\end{aligned}$$

□

Problem 2.

(a) Calculate the arc length functions $s(t) = \int_a^t \|\mathbf{f}'(u)\| \, du$ for the curves $\mathbf{f}(t) = (3\cos 2t, 3\sin 2t, 3t)$, for $0 \le t \le \pi/2$, and $\mathbf{g}(t) = (2\cos 3t, 2\sin 3t, 2t^{3/2})$ for $0 \le t \le 1$.

(b) Find the arc length parameterizations $\mathbf{f}(s)$ and $\mathbf{g}(s)$.

Solution.

(a) We have $\mathbf{f}'(t) = (-6\sin 2t, 6\cos 2t, 3)$ and $\|\mathbf{f}'(t)\| = \sqrt{36\sin^2 2t + 36\cos^2 2t + 9} = \sqrt{45} = 3\sqrt{5}$. Thus

$$s(t) = \int_0^t 3\sqrt{5}u \, du = 3\sqrt{5}t.$$

Inverting this, we have $t(s) = \frac{1}{3\sqrt{5}}s$, so the arc length parameterization of $\mathbf{f}$ is

$$\mathbf{f}(s) = \mathbf{f}(t(s)) = (3\cos(2/3\sqrt{5}s), 3\sin(2/3\sqrt{5}s), 1/\sqrt{5}s).$$

(b) For $\mathbf{g}$, we have $\mathbf{g}'(t) = (-6\sin 3t, 6\cos 3t, 3\sqrt{t})$, and $\|\mathbf{g}'(t)\| = \sqrt{36 + 9t} = 3\sqrt{4+t}$. Thus

$$s(t) = \int_0^t 3\sqrt{4+u} \, du = 2(4+t)^{3/2}.$$

Inverting this, we have $t(s) = (s/2)^{2/3} - 4$, so

$$\mathbf{g}(t(s)) = (2\cos(3(s/2)^{2/3} - 4), 2\sin(3(s/2)^{2/3} - 4), 2((s/2)^{2/3} - 4)^{3/2}).$$

□

Problem 3. If a curve $\mathbf{f}(s)$ is parameterized in terms of arc length, then $\frac{d}{ds}\|\mathbf{f}(s)\| = 1$, so the unit tangent becomes simply $\mathbf{T}(s) = \mathbf{f}'(s)$. The *curvature* of the curve is defined by

$$\kappa(s) = \left\|\frac{d\mathbf{T}(s)}{ds}\right\| = \left\|\frac{d^2\mathbf{f}(s)}{ds^2}\right\|.$$

Often, it is easier to compute using an arbitrary parameterization. Using the chain rule $\frac{d}{ds}\mathbf{T}(t(s)) = \frac{d}{dt}\mathbf{T}(t(s))\,\frac{dt}{ds}$ and the fact that $\frac{ds}{dt} = \|\mathbf{f}'(t)\|$, show that

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{f}'(t)\|}.$$

Show further that $\kappa(t)$ is given by

$$\kappa(t) = \frac{\|\mathbf{f}'(t) \times \mathbf{f}''(t)\|}{\|\mathbf{f}'(t)\|^3}.$$

Solution. Since $\frac{ds}{dt} = \|\mathbf{f}'(t)\|$, the inverse derivative $\frac{dt}{ds} = \frac{1}{\|\mathbf{f}'(t)\|}$, giving

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{f}'(t)\|}.$$

Using the identities proved in Problem 1, we have

$$\kappa(t) = \frac{\|\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t))\|}{\|\mathbf{f}'(t)\|^4} = \frac{\|\mathbf{f}'(t)\|\,\|(\mathbf{f}''(t) \times \mathbf{f}'(t))\|}{\|\mathbf{f}'(t)\|^4} = \frac{\|(\mathbf{f}''(t) \times \mathbf{f}'(t))\|}{\|\mathbf{f}'(t)\|^3},$$

again using the fact that $\mathbf{f}'(t)$ and $\mathbf{f}''(t) \times \mathbf{f}'(t)$ are orthogonal, so $\|\mathbf{f}'(t) \times (\mathbf{f}''(t) \times \mathbf{f}'(t))\| = \|\mathbf{f}'(t)\|\,\|\mathbf{f}''(t) \times \mathbf{f}'(t)\|$. □

Problem 4. Compute the curvature at each point for the helix $\mathbf{f}(t) = (\cos t, \sin t, t)$.

Solution. For the helix, from what what was computed in Problem 1, we have

$$\kappa(t) = \frac{\|\mathbf{f}'(t) \times \mathbf{f}''(t)\|}{\|\mathbf{f}'(t)\|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}.$$

□