Calculus III Workshop solutions: 8/30/17

Problem 1. A vector $\mathbf{v}$ in $\mathbb{R}^2$ lies in the positive quandrant (i.e., where $x \geq 0$ and $y \geq 0$), makes an angle of $\pi/3$ with the positive x-axis, and satisfies $\|\mathbf{v}\| = 4$. Write $\mathbf{v}$ in component form.

Solution. We're given that $\cos\theta = \pi/3$, where θ is the angle off the x-axis. The x component, v_1 of $\mathbf{v}$ is then given by $v_1 = \mathbf{v} \cdot \mathbf{i} = \|v\| \cos\theta = 4(\frac{1}{2}) = 2$. The y-component v_2 satisfies

$$4 = \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{4 + v_2^2}$$
$$\implies v_2 = \sqrt{12} = 2\sqrt{3}$$

Thus $\mathbf{v} = \big(2, 2\sqrt{3}\big) = 2\mathbf{i} + 2\sqrt{3}\mathbf{j}$. □

Problem 2. Find two unit vectors which are orthogonal to both of the specified vectors:
(a) $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$
(b) $(3, 2, 1)$ and $(-1, 1, 0)$.

Solution. (a) This can be done easily enough by hand: if $\mathbf{v} = (a, b, c)$ is orthogonal to the first vector, it must be that $0 = \mathbf{v} \cdot (\mathbf{i}+\mathbf{j}) = a + b$, which implies $a = -b$. Likewise, to be orthogonal to the second vector requires $0 = \mathbf{v} \cdot (\mathbf{i} + \mathbf{k}) = a + c$, so that $a = -c$. Then $b = c$, so two possibilities are $(1, -1, -1)$ and $(-1, 1, 1)$, which after normalization (i.e., dividing by the length to form unit vectors) become

$$\pm\left(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right).$$

Alternatively, you can use the cross product method.

(b) For this we use the cross product: calling the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, we have

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \mathbf{i}\big(2(0) - 1(1)\big) - \mathbf{j}\big(3(0) - 1(-1)\big) + \mathbf{k}\big(3(1) - (-1)2\big) = (-1, 1, 5)\,.$$

This vector is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ by a property of the cross product, but is not a unit vector: $\|\mathbf{a} \times \mathbf{b}\| = \sqrt{1 + 1 + 25} = \sqrt{27}$. The two unit vectors are therefore

$$\pm\tfrac{1}{\sqrt{27}}\,(-1, 1, 5)\,.$$

□

Problem 3. Let A, B and C be the vertices of a triangle in $\mathbb{R}^2$. Compute the vector $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

Solution. The sum is the zero vector. To see this, think in terms of displacement vectors. $\overrightarrow{AB}$ is the vector which, when added to the point A, gives the point B, etc. Thus the sum represents travelling from A, then to B, then to C, and back to A, which is a net displacement of 0. □

Problem 4. Find all vectors $\mathbf{v}$ such that $(1, 2, 1) \times \mathbf{v} = (3, 1, -5)$.

Solution. Let $\mathbf{v} = (v_1, v_2, v_3)$ have variable components. Writing the cross product, we have

$$(1,2,1) \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ v_1 & v_2 & v_3 \end{pmatrix} = (2v_3 - v_2, v_1 - v_3, v_2 - 2v_1)\,.$$

Setting this equal to $(3, 1, -5)$ gives the equations

$$\begin{aligned} 2v_3 - v_2 &= 3, \\ v_1 - v_3 &= 1, \\ v_2 - 2v_1 &= -5. \end{aligned}$$

The system has a free variable (say v_3, but this is not the only choice), in terms of which all solutions can be written as $(v_1, v_2, v_3) = (-1 + v3, -3 + 2v_3, v_3)$. □

Problem 5. Prove the following for all vectors $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$:
(a) $\|\mathbf{v} \times \mathbf{w}\|^2 + |\mathbf{v} \cdot \mathbf{w}|^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$
(b) If $\mathbf{v} \cdot \mathbf{w} = 0$ and $\mathbf{v} \times \mathbf{w} = \mathbf{0}$, then either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$.

Solution.
(a) Using the fact that the magnitude of $\mathbf{v} \times \mathbf{w}$ is $\|\mathbf{v}\| \|\mathbf{w}\| \sin\theta$ while $\mathbf{v} \cdot \mathbf{w} = \|\vec{v}\| \|\mathbf{w}\| \cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, we may square and add these to get

$$\|\mathbf{v} \times \mathbf{w}\|^2 + |\mathbf{v} \cdot \mathbf{w}|^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 (\sin^2\theta + \cos^2\theta) == \|\mathbf{v}\|^2 \|\mathbf{w}\|^2\,.$$

(b) Using the previous result, if both $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ and $\mathbf{v} \cdot \mathbf{w} = 0$, then

$$\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 = \|\mathbf{v} \times \mathbf{w}\|^2 + |\mathbf{v} \cdot \mathbf{w}|^2 = 0$$

which implies that either $\|\mathbf{v}\|$ or $\|\mathbf{w}\|$ vanishes; this in turn means that the corresponding vector is $\mathbf{0}$.

□

Problem 6. Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ all lie in the same plane in $\mathbb{R}^3$ if and only if $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

Solution. Recall that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ measures the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. If the three vectors all lie in the same plane, this parallelepiped collapses to a 1 or 2 dimensional object, so has volume 0. Conversely, if they do not lie in the same plane, then the parallelepiped must have some nonzero volume. □

Problem 7. Prove or give a counterexample: If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ for all vectors $\mathbf{u}$, then $\mathbf{v} = \mathbf{w}$.

Solution. First rerwrite the equation as $\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$, This is supposed to hold for all vectors $\mathbf{u}$, and in particular we can plug in $\mathbf{u} = \mathbf{v} - \mathbf{w}$, which yeilds

$$\|\mathbf{v} - \mathbf{w}\|^2 = 0,$$

so $\mathbf{v} - \mathbf{w}$ must be the zero vector, i.e., $\mathbf{v} = \mathbf{w}$. □