Calc III: Quiz 4 Solutions, Fall 2018

Problem 1. For the function $f(x, y, z) = x^2yz - xyz^3$,
(a) Find the gradient of f.
(b) Evaluate the gradient at the point $P(2, -1, 1)$.
(c) Find the directional derivative of f at P in the direction $\mathbf{v} = \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle$.

Solution. The gradient is

$$\nabla f(x, y, z) = \left\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2 \right\rangle .$$

Evaluated at P, we have

$$\nabla f(2, -1, 1) = \langle -4 + 1, 4 - 2, -4 + 6 \rangle = \langle -3, 2, 2 \rangle .$$

The directional derivative in the direction $\mathbf{v}$ (which is already a unit vector) is

$$D_{\mathbf{v}} f(2, -1, 1) = \nabla f(2, -1, 1) \cdot \mathbf{v} = \langle -3, 2, 2 \rangle \cdot \left\langle 0, \tfrac{4}{5}, -\tfrac{3}{5} \right\rangle = \frac{2}{5}.$$

$\square$

Problem 2. Find the critical points of the function $f(x, y) = x^2 + y^4 + 2xy$ and classify them as local maxima, minima or saddle points.

Solution. The critical points are given by $\nabla f = \mathbf{0}$, or

$$f_x = 2x + 2y = 0$$
$$f_y = 4y^3 + 2x = 0$$

We can simplify these to

$$x = -y$$
$$2y^3 = -x.$$

Plugging the first equation into the second gives the new pair

$$x = -y$$
$$2y^3 = y, \quad \text{or} \quad y(y^2 - \tfrac{1}{2}) = 0.$$

From the second equation either $y = 0$, $y = \frac{1}{\sqrt{2}}$, or $y = -\frac{1}{\sqrt{2}}$. In the first case, we get $x = 0$ from the first equation; in the second case we get $x = -\frac{1}{\sqrt{2}}$, and in the third case we get $x = \frac{1}{\sqrt{2}}$. Thus there are three critical points

$$(0, 0), \quad (\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}), \quad (-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}).$$

The discriminant for the second derivative test is

$$Df(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 6y^2 \end{vmatrix} = 12y^2 - 4.$$

We have

$$Df(0, 0) = -4 \implies (0, 0) \text{ is a saddle.}$$
$$Df(\pm\tfrac{1}{\sqrt{2}}, \mp\tfrac{1}{\sqrt{2}}) = 6 - 4 = 2, \quad f_{xx}(\pm\tfrac{1}{\sqrt{2}}, \mp\tfrac{1}{\sqrt{2}}) = 2 \implies (\pm\tfrac{1}{\sqrt{2}}, \mp\tfrac{1}{\sqrt{2}}) \text{ are minima.}$$

$\square$