Calc III: Quiz 5 Solutions, Fall 2018

Problem 1. Evaluate the integral $\iint_R (4-2y)\,dA$, where $R = [0,1]\times[0,1]$.

Solution.

$$\begin{aligned}\iint_R (4-2y)\,dA &= \int_0^1 \int_0^1 (4-2y)\,dx\,dy \\ &= \int_0^1 4x - 2xy\Big|_{x=0}^1 \,dy \\ &= \int_0^1 4-2y\,dy \\ &= 4y - y^2\Big|_{y=0}^1 \\ &= 3.\end{aligned}$$

□

Problem 2. Find the volume under the plane $3x+2y-z=0$ and above the region D enclosed by the parabolas $x=y^2$ and $y=x^2$.

Solution. The plane may be written as $z = f(x,y) = 3x+2y$, which is the integrand for our double integral. The region D between the parabolas can be written as either a Type I or Type II region, for instance

$$D = \left\{(x,y) : 0\le x\le 1,\ x^2\le y\le \sqrt{x}\right\}.$$

The volume is given by the double integral

$$\begin{aligned}\iint_D f(x,y)\,dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} 3x+2y\,dy\,dx \\ &= \int_0^1 3xy + y^2\Big|_{y=x^2}^{\sqrt{x}}\,dy \\ &= \int_0^1 (3x^{3/2} - 3x^3 + x - x^4)\,dy \\ &= \tfrac{6}{5}x^{5/2} - \tfrac{3}{4}x^4 + \tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\Big|_{x=0}^1 \\ &= \tfrac{6}{5} - \tfrac{3}{4} + \tfrac{1}{2} - \tfrac{1}{5} = \tfrac{3}{4}.\end{aligned}$$

□

Problem 3. Use polar coordinates to compute the integral $\iint_R \sin(x^2+y^2)\,dA$, where R is the region in the first quadrant ($x\ge 0$ and $y\ge 0$) between the circles centered at the origin with radii 1 and 3.

Solution. Using $r^2 = x^2+y^2$, the integrand becomes $\sin(r^2)$ in polar coordinates. The region of integration is a "polar rectangle" given by

$$R = \{1\le r\le 3,\ 0\le\theta\le\pi/2\}.$$

We recall that $dA = r\,dr\,d\theta$, so the integral in polar coordinates is

$$\begin{aligned}
\iint_R \sin(x^2+y^2)\,dA &= \int_0^{\pi/2}\int_1^3 \sin(r^2)r\,dr\,d\theta \\
&= \int_0^{\pi/2} -\tfrac{1}{2}\cos(r^2)\Big|_{r=1}^3\,d\theta \\
&= \int_0^{\pi/2} \tfrac{1}{2}(\cos(1)-\cos(9)\,d\theta \\
&= \tfrac{\pi}{4}(\cos(1)-\cos(9).
\end{aligned}$$

$\square$