Calc III: Quiz 6 Solutions, Fall 2018

Problem 1. A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant (where $x \geq 0$ and $y \geq 0$). Find its center of mass if the density at any point is equal to its distance to the x-axis.

Solution. The region is a polar rectangle given by $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$, with density $\delta(x, y) = y = r \sin\theta$. Thus the mass of the lamina is

$$\begin{aligned} M = \iint_L \delta \, dA &= \int_0^{\pi/2} \int_0^1 r \sin\theta \, r \, dr \, d\theta \\ &= \int_0^{\pi/2} \sin\theta \, d\theta \int_0^1 r^2 \, dr \\ &= \big(- \cos\theta \big) \Big|_{\theta=0}^{\pi/2} \big(\tfrac{r^3}{3} \big) \Big|_{r=0}^{1} \\ &= \frac{1}{3}. \end{aligned}$$

The center of mass is the point $(\overline{x}, \overline{y})$ where $\overline{x} = \frac{1}{M} \iint_L x \, \delta \, dA$ and $\overline{y} = \frac{1}{M} \iint_L y \, \delta \, dA$, so

$$\begin{aligned} \overline{x} &= \frac{1}{M} \int_0^{\pi/2} \int_0^1 (r\cos\theta)(r\sin\theta) \, r \, dr \, d\theta \\ &= 3 \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^1 r^3 \, dr \\ &= \frac{3}{2} \sin^2\theta \Big|_{\theta=0}^{\pi/2} \frac{r^4}{4} \Big|_{r=0}^{1} \\ &= \frac{3}{2} \frac{1}{4} = \frac{3}{8}. \end{aligned}$$

and

$$\begin{aligned} \overline{y} &= \frac{1}{M} \int_0^{\pi/2} \int_0^1 (r\sin\theta)^2 \, r \, dr \, d\theta \\ &= 3 \int_0^{\pi/2} \sin^2\theta \, d\theta \int_0^1 r^3 \, dr \\ &= \frac{3}{4} \int_0^{\pi/2} \frac{1 - \cos 2\theta}{2} \, d\theta \\ &= \frac{3}{4} \frac{\pi}{4} = \frac{3\pi}{16}. \end{aligned}$$

So the center of mass is $(\frac{3}{8}, \frac{3\pi}{16})$. □

Problem 2. Find the volume of the solid enclosed by the surface $y = x^2$ and the planes $z = 0$ and $y + z = 1$.

Solution. The limits in z go from 0 to $z = 1 - y$, over the region in the xy-plane bounded by $y = x^2$ and $y = 1$ (which is the intersection of the plane $y + z = 1$ with the xy-plane). So the integral is

$$
\begin{aligned}
\mathrm{Vol} &= \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} \, dz \, dy \, dx \\
&= \int_{-1}^{1} \int_{x^2}^{1} (1-y) \, dy \, dx \\
&= \int_{-1}^{1} -\tfrac{1}{2}(1-y)^2 \Big|_{y=x^2}^{1} \, dx \\
&= \tfrac{1}{2} \int_{-1}^{1} 1 - 2x^2 + x^4 \, dx \\
&= 1 - \tfrac{2}{3} + \tfrac{1}{5} \\
&= \frac{8}{15}.
\end{aligned}
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$\square$