Calc III: Quiz 8 Solutions, Fall 2018

Problem 1. For each of the following vector fields, determine whether $\mathbf{F}$ is conservative. If it is, find a function f such that $\nabla f = \mathbf{F}$.
(a) $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$
(b) $\mathbf{F}(x, y) = (xy + y^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$.

Solution.
(a) We test $P_y - Q_x = (e^x + \cos y) - (e^x + \cos y) = 0$, so $\mathbf{F}$ is conservative. A potential function f must satisfy the system of equations

$$f_x = ye^x + \sin y$$
$$f_y = e^x + x\cos y.$$

We can integrate the first equation in x to get

$$f = ye^x + x\sin y + c(y)$$

where $c(y)$ is an unknown function of y. Plugging this into the second equation gives

$$f_y = e^x + x\cos y + c'(y) = e^x + x\cos y$$

which implies $c'(y) = 0$ or $c(y) = k$, a constant, which we can take to be 0. Thus a potential function is given by $f(x, y) = ye^x + x\sin y$.
(b) We test $P_y - Q_x = (x + 2y) - (2x + xy) = -x \neq 0$, so $\mathbf{F}$ is not conservative.

□

Problem 2. Find a function f such that $\nabla f = \mathbf{F}$, where

$$\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$$

and use it to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

Solution. The vector field is given to be conservative, and a potential function f must satisfy

$$f_x = yz$$
$$f_y = xz$$
$$f_z = xy + 2z.$$

To solve by the "shuffle method", we integrate the equations in x, y, and z, respectively, giving

$$f = xyz + c_1(y, z)$$
$$f = xyz + c_2(x, z)$$
$$f = xyz + z^2 + c_3(x, y),$$

from which we see we can take $c_1(y, z) = c_2(x, z) = z^2$ and $c_3(x, y) = 0$, so $f(x, y, z) = xyz + z^2$ is a potential function.
By the Fundamental Theorem for Line Integrals, the line integral is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(4, 6, 3) - f(1, 0, -2) = 77.$$

□