Calc III: Quiz 9 Solutions, Fall 2018

Problem 1. Use Green's Theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y) = \langle y \cos x - xy \sin x,\ xy + x \cos x \rangle$$

and C is the closed triangular path from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ and back to $(0, 0)$.

Solution. Note that C has the opposite orientation from the convention required for Green's Theorem. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{\partial R} \mathbf{F} \cdot d\mathbf{r} = -\iint_R Q_x - P_y \, dA$$

where R is the solid triangle with the given vertices. We have

$$Q_x - P_y = (y + \cos x - x \sin x) - (\cos x - x \sin x) = y$$

so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\iint_R y \, dA = -\int_0^2 \int_0^{4-2x} y \, dy \, dx = -\frac{16}{3}.$$

□

Problem 2. For the vector field $\mathbf{F}(x, y, z) = xye^z \, \mathbf{i} + yze^x \, \mathbf{k}$,

(a) Compute the curl $\nabla \times \mathbf{F}$

Solution.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xye^z & 0 & yze^x \end{vmatrix} = \langle ze^x - 0,\ xye^z - yze^x,\ 0 - xe^z \rangle .$$

□

(b) Compute the divergence $\nabla \cdot \mathbf{F}$.

Solution.

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xye^z, 0, yze^x \rangle = ye^z + 0 + ye^x.$$

□

Problem 3. Evaluate the surface integral $\iint_S x^2 yz \, dS$, where S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$.

Solution. We use the graph parameterization $\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle$, where $z = g(x, y) = 1 + 2x + 3y$. Then

$$dS = |\mathbf{r}_x \times \mathbf{r}_y| \, dx \, dy = |\langle -g_x, -g_y, 1 \rangle| \, dx \, dy = \sqrt{g_x^2 + g_y^2 + 1} \, dx \, dy = \sqrt{4 + 9 + 1} \, dx \, dy = \sqrt{14} \, dx \, dy.$$

Thus

$$\begin{aligned}
\iint_S x^2yz\,dS &= \int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{14}\,dy\,dx \\
&= \sqrt{14}\int_0^3\int_0^2 x^2y+2x^3y+3x^2y^2\,dy\,dx \\
&= \sqrt{14}\int_0^3 2x^2+4x^3+8x^2\,dx \\
&= \sqrt{14}\left(2\tfrac{3^3}{3}+4\tfrac{3^4}{4}+8\tfrac{3^3}{3}\right) \\
&= 171\sqrt{14}.
\end{aligned}$$

□