Calc III: Workshop 10 Solutions, Fall 2018

Problem 1. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.

Solution. By Green's Theorem,

$$\oint_C \left\langle x^2 + 2y, 3x - y^2 \right\rangle \cdot d\mathbf{r} = \iint_D \tfrac{\partial x}{\partial (} 3x - y^2) - \tfrac{\partial y}{\partial (} x^2 + y)\, dA = \iint_D 3 - 1\, dA = 2\text{Area}(D) = 12.$$

$\square$

Problem 2. Let D be a region bounded by a simple closed curve C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid (i.e., center of mass assuming uniform density) $(\overline{x}, \overline{y})$ of D are

$$\begin{aligned}\overline{x} &= \frac{1}{2A}\oint_C x^2\, dy = \frac{1}{2A}\oint_C 0\, dx + x^2\, dy,\\ \overline{y} &= -\frac{1}{2A}\oint_C y^2\, dx = -\frac{1}{2A}\oint_C y^2\, dx + 0\, dy,\end{aligned}$$

where A is the area of D.

Solution. By Green's Theorem,

$$\frac{1}{2A}\oint_C x^2\, dy = \frac{1}{2A}\iint_D 2x\, dA = \frac{1}{A}\iint_D x\, dA = \overline{x}.$$

Similarly,

$$-\frac{1}{2A}\oint_C y^2\, dx = \frac{1}{2A}\iint_D 2y\, dA = \frac{1}{A}\iint_D y\, dA = \overline{y}.$$

$\square$

Problem 3. Use the previous exercise to show that the centroid of a quarter-circular region of radius a.

Solution. We can compute the coordinates of the centroid as a line integral, for example:

$$\overline{x} = \frac{1}{2A}\oint_C x^2\, dy.$$

The closed curve C has three parts: C_1, the line segment along the x-axis from $(0,0)$ to $(a,0)$, C_2, the circular arc from $(a,0)$ to $(0,a)$, and C_3, the line segment along the y-axis from $(a,0)$ to $(0,0)$. Along C_1 and C_3, the line integral vanishes, since in the first case $dy = 0$ and in the second case $x^2 = 0$. Thus

$$\begin{aligned}\overline{x} &= \frac{1}{2A}\int_0^{\pi/2} a^2 \cos^2\theta\, da \sin\theta\\ &= \frac{1}{2}\frac{4}{\pi a^2}\int_0^{\pi/2} a^3 \cos^3\theta\, d\theta\\ &= \frac{2a}{\pi}\int_0^{\pi/2} \cos\theta(1 - \sin^2\theta)\, d\theta\\ &= \frac{4a}{3\pi}.\end{aligned}$$

By symmetry $\overline{y} = \overline{x}$. $\square$

Problem 4. Use Green's Theorem to evaluate the line integral $\int_C \langle y+1, x\rangle \cdot d\mathbf{r}$, where C is the upper half of the unit circle starting at $(1,0)$ and ending at $(-1,0)$. Note that C is not closed!

Solution. To use Green's Theorem, we need a closed curve, so the simplest way to close the given curve is to add the line segment C' from $(-1,0)$ to $(1,0)$. Then

$$\oint_{C+C'} \langle y+1, x\rangle \cdot d\mathbf{r} = \int_C \langle y+1, x\rangle \cdot d\mathbf{r} + \int_{C'} \langle y+1, x\rangle \cdot d\mathbf{r} = \iint_R 1 - 1\, dA = 0.$$

Thus

$$\int_C \langle y+1, x\rangle \cdot d\mathbf{r} = -\int_{C'} \langle y+1, x\rangle \cdot d\mathbf{r}.$$

Thus we only need to integrate the line integral along the straight line path from $(-1,0)$ to $(1,0)$. Using $\mathbf{r}(t) = \langle t, 0\rangle$, $-1 \le t \le 1$, we have

$$-\int_{C'} \langle y+1, x\rangle \cdot d\mathbf{r} = -\int_{-1}^{1} \langle 0+1, t\rangle \cdot \langle 1, 0\rangle \; dt = -\int_{-1}^{1} dt = -2.$$

Of course the vector field is actually conservative in this example, so we could as well have found a potential function, say $f(x,y) = xy + x$, and used the Fundamental Theorem for Line Integrals. $\square$

Problem 5.

(a) Show that for any vector field $\mathbf{F}(x,y,z)$ with twice continuously differentiable components, that

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

(b) Is there a vector field $\mathbf{G}$ such that $\nabla \times \mathbf{G} = \langle x\sin y, \cos y, z - xy\rangle$? Explain.

Solution.

(a) If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then

$$\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

since mixed second partial derivatives can be taken in either order.

(b) No: since $\nabla \cdot \mathbf{G} = \sin y - \sin y + 1 = 1 \neq 0$, this is not possible by the previous result.

$\square$

Problem 6.

(a) Show that any vector field of the form

$$\mathbf{F}(x,y,z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$$

is irrotational ($\nabla \times \mathbf{F} = \mathbf{0}$).

(b) Show that any vector field of the form

$$\mathbf{F}(x,y,z) = f(y,z)\mathbf{i} + g(x,z)\mathbf{j} + h(x,y)\mathbf{k}$$

is incompressible ($\nabla \cdot \mathbf{F} = 0$).

Solution.

(a)
$$\nabla \times \mathbf{F} = \left(\tfrac{\partial y}{\partial h}(z) - \tfrac{\partial z}{\partial g}(y)\right)\mathbf{i} + \left(\tfrac{\partial z}{\partial f}(x) - \tfrac{\partial z}{\partial h}(z)\right)\mathbf{j} + \left(\tfrac{\partial x}{\partial g}(y) - \tfrac{\partial y}{\partial f}(x)\right)\mathbf{k} = \mathbf{0},$$
so $\mathbf{F}$ is irrotational.

(b)
$$\nabla \cdot \mathbf{F} = \tfrac{\partial x}{\partial f}(y,z) + \tfrac{\partial y}{\partial g}(x,z) + \tfrac{\partial z}{\partial h}(x,y) = 0,$$
so $\mathbf{F}$ is incompressible.

$\square$

Problem 7. Find a parametric representation for the following surfaces:

(a) The plane that passes through the point $(0,-1,5)$ and contains the vectors $\langle 2,1,4\rangle$ and $\langle -3,2,5\rangle$.

(b) The part of the ellipsoid $x^2+2y^2+3z^2=1$ that lies to the left of the xz-plane.

Solution.

(a) A simple one is to express the plane as
$$\mathbf{r}(u,v) = \langle 0,-1,5\rangle + u\,\langle 2,1,4\rangle + v\,\langle -3,2,5\rangle = \langle 2u-3v,\ -1+u+2v,\ 5+4u+5v\rangle\,,$$
where the domain for (u,v) is all of $\mathbb{R}^2$.

(b) The best way is to view this as the "sideways graph" of $y = -\frac{1}{\sqrt{2}}\sqrt{1-x^2-3z^2}$, and use the parameterization
$$\mathbf{r}(x,z) = \left\langle x,\ -\tfrac{1}{\sqrt{2}}\sqrt{1-x^2-3z^2},\ z\right\rangle$$
with (x,z) in the ellipse $x^2+3z^2 \le 1$.

$\square$