Calc III: Workshop 11 Solutions, Fall 2018

Problem 1. The *helicoid* or "spiral ramp" is a surface parameterized by $\mathbf{r}(u, v) = \langle u\cos v,\, u\sin v,\, v\rangle$, say for $0 \le v \le 2\pi$ and $0 \le u \le 1$. See if you can sketch a graph of this surface.
(a) Find the tangent plane to the helicoid at the point $(1, 0, 0)$.
(b) Set up an integral which computes its surface area. (You do not have to evaluate it!)

Solution.
(a) A (not-necessarily unit) normal vector to the helicoid at any point $\mathbf{r}(u, v)$ is given by

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \langle \cos v,\, \sin v,\, 0\rangle \times \langle -u\sin v,\, u\cos v,\, 1\rangle = \langle \sin v,\, -\cos v,\, u\rangle$$

At the point $(1, 0, 0) = \mathbf{r}(1, 0)$, we have

$$\mathbf{n} = \langle 0, -1, 1\rangle$$

so the tangent plane is given by the equation

$$0 = \mathbf{n}\cdot\langle x-1, y-0, z-0\rangle = -y + z = 0.$$

(b) The surface area element is

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \sqrt{\sin^2 v + \cos^2 v + u^2}\, du\, dv = \sqrt{1+u^2}\, du\, dv$$

so the surface area is

$$\begin{aligned}
\text{Area} = \iint_S dS &= \int_0^{2\pi}\int_0^1 \sqrt{1+u^2}\, du\, dv \\
&= 2\pi \int_0^1 \sqrt{1+u^2}\, du \\
&= 2\pi \int_0^{\pi/4} \sqrt{1+\tan^2\varphi}\, \sec^2\varphi\, d\varphi \\
&= 2\pi \int_0^{\pi/4} \sec^3\varphi\, d\varphi \\
&= \cdots \text{using integration by parts} \cdots \\
&= 2\pi \left(\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left|u + \sqrt{1+u^2}\right|\right)\Big|_{u=0}^{1} \\
&= \pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right).
\end{aligned}$$

$\square$

Problem 2. Find the surface area of the part of the plane $x + 2y + 3z = 1$ which lies inside the cylinder $x^2 + y^2 = 3$.

Solution. Solving for z in the equation for the plane, we have $z = \frac{1}{3}(1 - x - 2y)$. We can use x and y as parameters, with parameterization

$$\mathbf{r}(x, y) = \left\langle x, y, \tfrac{1}{3}(1 - x - 2y)\right\rangle$$

where (x, y) vary in the disk R of radius 3. Then

$$\mathbf{r}_x(x, y) = \left\langle 1, 0, -\tfrac{1}{3}\right\rangle, \quad \mathbf{r}_y(x, y) = \left\langle 0, 1, -\tfrac{2}{3}\right\rangle$$

and

$$dS = |\mathbf{r}_x \times \mathbf{r}_y|\, dx\, dy = \sqrt{(\tfrac{1}{3})^2 + (\tfrac{2}{3})^2 + 1}\, dx\, dy = \frac{\sqrt{14}}{3}\, dx\, dy.$$

Thus the area is

$$\iint_S 1\, dS = \iint_R \frac{\sqrt{14}}{3}\, dx\, dy = \frac{\sqrt{14}}{3}\text{Area}(R) = \frac{\sqrt{14}}{3}(3\pi) = \pi\sqrt{14},$$

since R is the disk of radius $\sqrt{3}$. □

Problem 3. Find the surface area of the part of the cone $z = \sqrt{x^2+y^2}$ between $z = 0$ and $z = H$.

Solution. We can use x and y as paramters, with $\mathbf{r}(x, y) = \left\langle x, y, \sqrt{x^2+y^2} \right\rangle$ and (x, y) varying in the disk of radius H, or we can use polar/cylindrical coordinates directly, with parameterization

$$\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle\,, \quad 0 \le \theta \le 2\pi,\ 0 \le r \le H,$$

using the fact that $z = r$ on the cone. Using the latter parameterization, we find

$$\mathbf{r}_r(r, \theta) = \langle \cos\theta, \sin\theta, 1\rangle\,, \quad \mathbf{r}_\theta(r, \theta) = \langle -r\sin\theta, r\cos\theta, 0\rangle\,, \quad \mathbf{r}_r \times \mathbf{r}_\theta = \langle -r\cos\theta, -r\sin\theta, r\rangle\,,$$

so

$$dS = |\mathbf{r}_r \times \mathbf{r}_\theta|\; dr\, d\theta = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}\, dr\, d\theta = \sqrt{2}\, r\, dr\, d\theta.$$

The surface area is given by

$$\text{Area}(S) = \iint_S dS = \int_0^{2\pi} \int_0^H \sqrt{2}\, r\, dr\, d\theta = (\sqrt{2})(2\pi)(\tfrac{H^2}{2}) = \sqrt{2}\pi H^2.$$

□

Problem 4. Compute the flux $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ of the vector field $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ lying over the square $0 \le x \le 1$, $0 \le y \le 1$ and has upward orientation.

Solution. Given the limits $0 \le x \le 1$ and $0 \le y \le 1$, it is best to parameterize S by x and y here, so

$$\mathbf{r}(x, y) = \left\langle x, y, 4 - x^2 - y^2 \right\rangle, \quad \mathbf{r}_x(x, y) = \langle 1, 0, -2x\rangle\,, \quad \mathbf{r}_y(x, y) = \langle 0, 1, -2y\rangle\,, \quad \mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1\rangle\,.$$

Then

$$\mathbf{n}\, dS = \pm\mathbf{r}_x \times \mathbf{r}_y\, dx\, dy$$

with the $\pm$ sign determined by the orientation. Since we want $\mathbf{n}$ to point "upward" and $\mathbf{r}_x \times \mathbf{r}_y$ has positive $\mathbf{k}$ component, we take the $+$ sign. So

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \int_0^1 \int_0^1 \left\langle xy, y(4-x^2-y^2), (4-x^2-y^2)x \right\rangle \cdot \langle 2x, 2y, 1\rangle \; dx\, dy \\
&= \int_0^1 \int_0^1 2x^2y + 2y^2(4-x^2-y^2) + (4-x^2-y^2)x \, dx\, dy \\
&= \int_0^1 \int_0^1 2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2 \, dx\, dy \\
&= \int_0^1 \frac{2}{3}y + 8y^2 - \frac{2}{3}y^2 - 2y^4 + 2 - \frac{1}{4} - \frac{y^2}{2} \, dy \\
&= \frac{2}{6} + \frac{8}{3} - \frac{2}{9} - \frac{2}{5} + 2 - \frac{1}{4} - \frac{1}{6} \\
&= \frac{713}{180}
\end{aligned}
$$

□

Problem 5. Compute the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ of the vector field $\mathbf{F}(x,y,z) = -x\mathbf{i} - y\mathbf{j} + z^3\mathbf{k}$ where S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 3$ with downward orientation.

Solution. A good choice for parameterization of S is in terms of cylindrical/polar coordinates: $z = \sqrt{x^2+y^2} = r$, so we can take

$$\mathbf{r}(r,\theta) = \langle r\cos\theta,\, r\sin\theta,\, r\rangle\,, \quad 1 \le r \le 3, \quad 0 \le \theta \le 2\pi.$$

Then

$$\mathbf{r}_r = \langle \cos\theta,\, \sin\theta,\, 1\rangle\,, \quad \mathbf{r}_\theta = \langle -r\sin\theta,\, r\cos\theta,\, 0\rangle\,, \quad \mathbf{r}_r \times \mathbf{r}_\theta = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

and since we want the downward orientation we should take $-\mathbf{r}_r \times \mathbf{r}_\theta$ in order to get a negative $\mathbf{k}$ component in our normal vector. Thus

$$\mathbf{n}\, dS = \langle r\cos\theta, r\sin\theta, r\rangle \; dr\, d\theta.$$

Since

$$\mathbf{F}\big(\mathbf{r}(r,\theta)\big) = -r\cos\theta, -r\sin\theta, r^3,$$

we have

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \int_0^{2\pi} \int_1^3 \left\langle -r\cos\theta, -r\sin\theta, r^3 \right\rangle \cdot \langle r\cos\theta, r\sin\theta, r\rangle \; dr\, d\theta \\
&= \int_0^{2\pi} \int_1^3 r^4 - r^2 dr\, d\theta \\
&= 2\pi \left(\tfrac{3^5}{5} - \tfrac{1}{5} - \tfrac{3^3}{3} + \tfrac{1}{3} \right) \\
&= \frac{1192\pi}{15}.
\end{aligned}
$$

□