Calc III: Workshop 2 Solutions, Fall 2018

Problem 1. Determine whether the following planes are parallel, perpindicular, or neither. If neither, find the angle between them.

(a) $9x - 3y + 6z = 2$ and $2y = 6x + 4z$.
(b) $x - y + 3z = 1$ and $3x + y - z = 2$.

Solution.

(a) We extract normal vectors for the planes as the coefficients of x, y, and z in each equation, giving

$$\mathbf{n}_1 = \langle 9, -3, 3 \rangle, \quad \text{and} \quad \mathbf{n}_2 = \langle 6, -2, 4 \rangle$$

for the two planes, respectively. These vectors are parallel, since $\mathbf{n}_2 = \frac{2}{3}\mathbf{n}_1$, hence the planes themselves are parallel.

(b) Two normal vectors are $\mathbf{n}_1 = \langle 1, -1, 3 \rangle$ and $\mathbf{n}_2 = \langle 3, 1, -1 \rangle$, respectively, which are not multiples of each other, so the planes are not parallel. We have

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (1)(3) + (-1)(1) + (3)(-1) = -1$$

so the vectors (and hence the planes) are not orthogonal. So the planes are neither parallel nor orthogonal.

$\square$

Problem 2. Find the line of intersection of the planes

$$x + 3y + 2z - 6 = 0, \quad 2x - y + z + 2 = 0.$$

Solution. The planes have normal vectors $\mathbf{n}_1 = (1, 3, 2)$ and $\mathbf{n}_2 = (2, -1, 1)$, respectively. Since these are not parallel, the two planes must intersect, and the resulting line will be parallel to $\mathbf{n}_1 \times \mathbf{n}_2 = (5, 3, -7)$. It remains to find any single point in their intersection. Requiring both equations above to hold, we can set $x = 0$ (for instance), to get the system of equations

$$\begin{aligned} 3y + 2z &= 6, \\ y &= z + 2. \end{aligned}$$

The second is easily substituted into the first to get $z = 0$, from which we then have $y = 2$. Thus $(0, 2, 0)$ is a point on the line, and we can write a parameterized equation for the line as

$$\begin{aligned} x &= 0 + 5t, \\ y &= 2 + 3t, \\ z &= 0 - 7t. \end{aligned}$$

$\square$

Problem 3. Find the point of intersection (if any) of the line $\frac{x-6}{4} = y + 3 = z$ with the plane $x + 3y + 2z - 6 = 0$.

Solution. Plugging the equations for the line into the equation for the plane to eliminate y and z, we have

$$x + 3\left(\frac{x-6}{4} - 3\right) + 2\left(\frac{x-6}{4}\right) - 6 = 0$$

which simplifies to $x = 10$. Plugging this into the equation for the line gives the point $(10, -2, 1)$. □

Problem 4. Find an equation for the surface obtained by rotating the line $z = 3y$, $x = 0$, about the z-axis.

Solution. Proceding graphically, we see that the surface is a cone which opens along the z-axis. The general equation for such a cone is $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Since we must recover the line $z = 3y$ when $x = 0$, it follows that $b = \frac{1}{3}$. By symmetry (i.e., when we have rotated the line into the xz-plane), we must also have that $a = \frac{1}{3}$, so the equation of the surface is

$$z^2 = 9(x^2 + y^2).$$

□

Problem 5. Reduce the equation to one of the standard forms, classify the surface, and sketch it.
(a) $x^2 + y^2 - 2x - 6y - z + 10 = 0$.
(b) $x^2 - y^2 + z^2 - 4x - 2z + 3 = 0$.
(c) $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$.

Solution. To put these equations in standard form, we need to complete the squares to get rid of the linear terms in x, y, and z.

(a) Completing the square, we have

$$(x-1)^2 + (y-3)^2 - z + 10 - 1 - 9 = (x-1)^2 + (y-3)^2 - z = 0$$

For standard form, we would write

$$z = (x-1)^2 + (y-3)^2.$$

This is an upward opening elliptic paraboloid, with its base at the point $(1, 3, 0)$.

(b) Completing the square, we have

$$(x-2)^2 - y^2 + (z-1)^2 + 3 - 4 - 1 = (x-2)^2 - y^2 + (z-1)^2 + -2 = 0$$

Rearranging, we would write this in standard form as

$$1 = \frac{(x-2)^2}{2} - \frac{y^2}{2} + \frac{(z-1)^2}{2}$$

This is a 1-sheeted hyperboloid, with central axis the line $x = 2$, $z = 1$ (parallel to the y-axis).

(c) Completing the square, we have

$$4(x-3)^2 + (y-4)^2 + (z+2)^2 + 55 - 36 - 16 - 4 = 4(x-3)^2 + (y-4)^2 + (z+2)^2 - 1 = 0$$

In standard form, we have

$$1 = \frac{(x-3)^2}{(1/2)^2} + (y-4)^2 + (z+2)^2,$$

which is an ellipsoid centered at $(3, 4, -2)$, with radii $1/2$ (along the x axis), 1 and 1 (along the y and z axes), respectively.

$\square$

Problem 6. In general, any four non-coplanar points determine a unique sphere. Find the equation for the sphere determined by the points $(0, 0, 0)$, $(0, 0, 2)$, $(1, -4, 3)$, and $(0, -1, 3)$.

Solution. Plug these into the general form $x^2 + y^2 + z^2 + ax + by + cz + d = 0$ to get the system of equations

$$\begin{aligned} d &= 0, \\ 2c + d &= -4, \\ a - 4b + 3c + d &= -26, \\ -b + 3c + d &= -10 \end{aligned}$$

These can be solved by substitution to get $a = -4$, $b = 4$, $c = -2$ and $d = 0$. Completing the square and rewriting the equation in the form $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$, we find that the center of the sphere is $(x_0, y_0, z_0) = (-a/2, -b/2, -c/2) = (2, -2, 1)$ and the radius is $r = \sqrt{\frac{1}{4}(a^2 + b^2 + c^2) - d} = 3$. $\square$