Calc III: Workshop 3 Solutions, Fall 2018

Problem 1. Let C be the curve with equations $x = 2 - t^3$, $y = 2t - 1$, and $z = \ln t$. Find
(a) the point where C intersects the xz-plane, and
(b) parametric equations for the tangent line to C at the point $(1, 1, 0)$.

Solution.
(a) C intersects the xz-plane when

$$0 = y = 2t - 1 \implies t = \tfrac{1}{2}.$$

Plugging this back in gives the point

$$(x, y, z) = (\tfrac{15}{8}, 0, \ln \tfrac{1}{2})$$

(b) The vector equation for the curve is $\mathbf{r}(t) = \langle 2 - t^3, 2t - 1, \ln t \rangle$. The point in question is $\mathbf{r}(1) = (1, 1, 0)$, corresponding to parameter value $t = 1$. The derivative is given by

$$\mathbf{r}'(t) = \left\langle -2t^2, 2, \tfrac{1}{t} \right\rangle,$$

so at the point $(1, 1, 0)$ we have the tangent vector

$$\mathbf{r}'(1) = \langle -2, 2, 1 \rangle.$$

The vector equation for the tangent line is therefore

$$\ell(s) = \langle 1, 1, 0 \rangle + s \langle -2, 2, 1 \rangle = \langle 1 - 2s, 1 + 2s, s \rangle,$$

or in parametric form

$$x(s) = 1 - 2s, \quad y(s) = 1 + 2s, \quad z(s) = s.$$

□

Problem 2. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$.

Solution. The problem is to find $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ which satisfy $x(t)^2 + y(t)^2 = 16$ and $x(t) + z(t) = 5$. There are multiple possible solutions. We can satisfy the first equation by

$$x(t) = 4\cos(t) \quad \text{and} \quad y(t) = 4\sin(t),$$

and plugging this into the second equation gives

$$4\cos(t) + z(t) = 5 \qquad \implies \qquad z(t) = 5 - 4\cos(t).$$

So one solution is given by

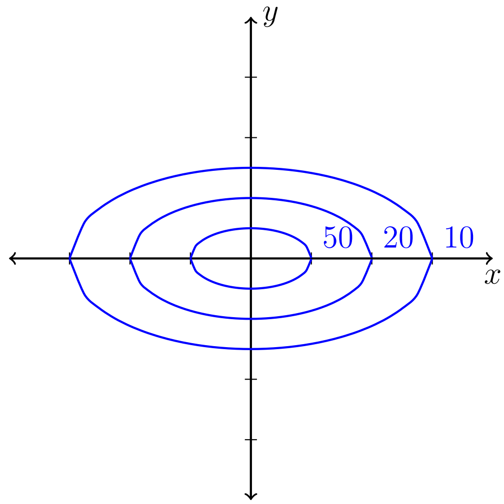
$$\mathbf{r}(t) = \langle 4\cos(t),\, 4\sin(t),\, 5 - 4\cos(t) \rangle.$$

□

Problem 3. A thin metal plate, located in the xy-plane, has temperature $T(x, y)$ at the point (x, y). Sketch some level curves (isothermals) if the temperature function is given by

$$T(x, y) = \frac{100}{1 + x^2 + 2y^2}.$$

Solution. The level curves are concentric ellipses:



$\square$

Problem 4. Describe the level *surfaces* of the 3 variable functions
(a) $f(x,y,z) = x^2 + 3y^2 + 5z^2$,
(b) $f(x,y,z) = y^2 + z^2$.

Solution.
(a) The level surfaces are concentric ellipsoids centered at the origin, with longest axes in the x direction and shortest axes in the z direction. The values of f along the level surfaces decrease down to 0 as we approach $(0,0,0)$ and increase as we go away from $(0,0,0)$.
(b) The level surfaces are concentric cylinders, centered on the x-axis, with circular profile. The values of f along the level surfaces decrease down to 0 as the cylinders get smaller (as we approach the x-axis), and increase for larger cylinders (as we move away from the x-axis).

$\square$

Problem 5. Find the limit, if it exists, or show the limit does not exist:
(a) $\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$
(b) $\lim_{(x,y)\to(0,0)} \dfrac{x^4 - 4y^2}{x^2 + 2y^2}$
(c) (Optional bonus) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

Solution.
(a) The function is continuous, as it is made up of sums and products of continuous functions, so the limit is just the value of the function at $(3,2)$, or

$$\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2) = (3)^2(2)^3 - 4(2)^2 = 56.$$

(b) Along the x-axis ($x = t$, $y = 0$), the pathwise limit is given by

$$\lim_{t\to 0} \frac{t^4 - 0}{t^2 + 0} = \lim_{t\to 0} t^2 = 0.$$

On the other hand, along the y-axis ($x = 0$, $y = t$), the pathwise limit is given by

$$\lim_{t\to 0} \frac{0 - 4t^2}{0 + 2t^2} = \lim_{t\to 0} -2 = -2.$$

Since these values do not agree, the limit does not exist.

(c) The pathwise limits along the x and y-axes are given by

$$\lim_{t\to 0} \frac{t(0)}{\sqrt{t^2 + 0}} = 0, \quad \text{and} \quad \lim_{t\to 0} \frac{(0)t}{\sqrt{0 + t^2}} = 0,$$

respectively, and along any line $y = cx$ ($x = t$, $y = ct$), we have

$$\lim_{t\to 0} \frac{ct^2}{\sqrt{t^2 + c^2t^2}} = \lim_{t\to 0} \frac{ct^2}{\sqrt{1 + c^2t}} = \lim_{t\to 0} \frac{c}{\sqrt{1 + c^2}}t = 0.$$

so we begin to suspect the limit exists and equals 0. To show this properly we note that

$$\left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right| = \frac{|x|\,|y|}{\sqrt{x^2 + y^2}} \le \frac{(\sqrt{x^2 + y^2})(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2}$$

since both $|x| \le \sqrt{x^2 + y^2}$ and $|y| \le \sqrt{x^2 + y^2}$. Thus given any $\varepsilon > 0$, we can set $\delta = \varepsilon$ and then whenever

$$|\langle x, y\rangle - \langle 0, 0\rangle| = \sqrt{x^2 + y^2} < \delta$$

it follows that

$$\left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right| \le \sqrt{x^2 + y^2} < \delta = \varepsilon,$$

so the definition of the limit is satisfied.

□

Problem 6. Suppose $f(t)$ and $g(t)$ are single variable functions with ordinary derivatives $f'(t)$ and $g'(t)$, respectively. Compute the partial derivatives $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$, where

(a) $h(x, y) = f(x) + g(y)$
(b) $h(x, y) = f(x)g(y)$
(c) $h(x, y) = f(x + y)$
(d) $h(x, y) = f(xy)$
(e) $h(x, y) = f(x/y)$

Solution.

(a) $\frac{\partial h}{\partial x} = f'(x)$, $\frac{\partial h}{\partial y} = g'(y)$.
(b) $\frac{\partial h}{\partial x} = g(y)f'(x)$, $\frac{\partial h}{\partial y} = f(x)g'(y)$.
(c) $\frac{\partial h}{\partial x} = f'(x + y)$, $\frac{\partial h}{\partial y} = f'(x + y)$.
(d) $\frac{\partial h}{\partial x} = yf'(xy)$, $\frac{\partial h}{\partial y} = xf'(xy)$.
(e) $\frac{\partial h}{\partial x} = \frac{1}{y}f'(x/y)$, $\frac{\partial h}{\partial y} = -\frac{x}{y^2}f'(x/y)$.

□

Problem 7. The *diffusion equation* or *heat equation*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where $k > 0$ is a constant, models the diffusion of heat $u(x,t)$ through a thin wire ($x =$ location along the wire, $t =$ time), or the concentration $u(x,t)$ of a pollutant at time t at a distance x from the source of the pollution. Verify that the function

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}$$

is a solution to the diffusion equation.

Solution. Computing the partial derivatives with respect to x gives

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{\sqrt{4\pi kt}}\left(\frac{-2x}{4kt}\right)e^{-x^2/(4kt)} = \frac{-x}{2kt}u(x,t)\\
\frac{\partial^2 u}{\partial x^2} &= -\frac{1}{2kt}u(x,t) - \frac{x}{2kt}\frac{\partial u}{\partial x}\\
&= \left(-\frac{1}{2kt} + \frac{x^2}{(2kt)^2}\right)u(x,t),
\end{aligned}$$

while the partial derivative with respect to t is

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{1}{2}\left(\frac{4\pi k}{(4\pi kt)^{3/2}}\right)e^{-x^2/(4kt)} + \frac{1}{\sqrt{4\pi kt}}\left(\frac{x^2(4k)}{(4kt)^2}\right)e^{-x^2/(4kt)}\\
&= \left(\frac{-k}{2kt} + \frac{kx^2}{(2kt)^2}\right)\frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}\\
&= k\left(\frac{x^2}{(2kt)^2} - \frac{1}{2kt}\right)u(x,t),
\end{aligned}$$

so indeed $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ holds. $\square$