Calc III: Workshop 4 Solutions, Fall 2018

Problem 1. Let $u = e^{r\theta}\sin\theta$. Compute the partial derivative $\frac{\partial^3 u}{\partial r^2 \partial\theta}$.

Solution. We have

$$\begin{aligned} u_r &= \theta e^{r\theta}\sin\theta \\ u_{rr} &= \theta^2 e^{r\theta}\sin\theta, u_{rr\theta} \quad = 2\theta e^{r\theta}\sin\theta + \theta^2 r e^{r\theta}\sin\theta + \theta^2 e^{r\theta}\cos\theta. \end{aligned}$$

□

Problem 2. Determine whether each of the following functions is a solution to *Laplace's equation* $u_{xx} + u_{yy} = 0$.

(a) $u = x^2 + y^2$
(b) $u = x^2 - y^2$
(c) $u = x^3 - 3xy^2$
(d) $u = \ln\sqrt{x^2 + y^2}$

Solution.

(a) $u_{xx} = 2$, $u_{yy} = 2$, so $u_{xx} + u_{yy} = 4 \neq 0$.
(b) $u_{xx} = 2$, $u_{yy} = -2$, so $u_{xx} + u_{yy} = 0$.
(c) $u_{xx} = 6x$, $u_{yy} = -6x$ so $u_{xx} + u_{yy} = 0$.
(d)

$$\begin{aligned} u_x &= \frac{x}{x^2+y^2}, & u_y &= \frac{y}{x^2+y^2}, \\ u_{xx} &= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \\ u_{yy} &= \frac{x^2 - y^2}{(x^2+y^2)^2} \end{aligned}$$

so $u_{xx} + u_{yy} = 0$.

□

Problem 3. Is it possible that a function $f(x, y)$ has partial derivatives $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$?

Solution. No. Recall that for a nice function (having continuous second partial derivatives), $f_{xy} = f_{yx}$. Since $\frac{\partial}{\partial y} f_x = 4$ and $\frac{\partial}{\partial x} f_y = 3$ are continuous but not equal, there cannot be such a function. □

Problem 4. Find the tangent plane at $(2, -1, -3)$ to the surface

$$z = 3y^2 - 2x^2 + x$$

Solution. The linear approximation of $f(x, y) = 3y^2 - 2x^2 + x$ at $(x_0, y_0) = (2, -1)$ is given by

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= -3 + (-2(2) - 1)(x - 2) + (3(-1))(y + 1) \\ &= -3 - 5(x - 2) - 3(y + 1). \end{aligned}$$

The tangent plane is then the graph $z = L(x, y)$ of this linear approximation, so

$$z = -3 - 5(x - 2) - 3(y + 1).$$

$\square$

Problem 5. Find the tangent plane at $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ to the surface

$$x^2 + 2y^2 + z^2 = 1.$$

Solution. The surface is not in the form $z = f(x, y)$, but we can write it in this form by solving for z:

$$z = -\sqrt{1 - x^2 - 2y^2}.$$

Note that we take the negative square root since the point of interest is below the xy-plane. The partial derivatives of $f(x, y) = -\sqrt{1 - x^2 - 2y^2}$ are given by

$$f_x(x, y) = \frac{x}{\sqrt{1 - x^2 - 2y^2}}, \qquad f_y(x, y) = \frac{2y}{\sqrt{1 - x^2 - 2y^2}},$$

so

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = -\tfrac{1}{2} + (1)(x - \tfrac{1}{2}) + (2)(y - \tfrac{1}{2})$$

and the tangent plane is given by

$$z = L(x, y) == -\tfrac{1}{2} + (1)(x - \tfrac{1}{2}) + (2)(y - \tfrac{1}{2}).$$

$\square$

Problem 6. The length ℓ, width w and height h of a box change with time. At a certain instant the dimensions are $\ell = 1$ m and $w = h = 2$ m, and ℓ and w are increasing at a rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that instant find the rates at which the following quantities are changing:

(a) The volume
(b) The surface area
(c) The length of a diagonal

Solution.

(a) The volume, as a function of ℓ, w and h is

$$V(\ell, w, h) = \ell w h$$

so

$$\begin{aligned}\frac{d}{dt}V &= w(t)h(t)\ell'(t) + \ell(t)h(t)w'(t) + \ell(t)w(t)h'(t) \\ &= (2)(2)(2) + (1)(2)(2) + (1)(2)(-3) = 6\text{m}^3/\text{s}.\end{aligned}$$

(b) The surface area is given by

$$S(\ell, w, h) = 2\ell w + 2\ell h + 2wh$$

so

$$\frac{d}{dt}S = 2(w+h)\ell' + 2(\ell+h)w' + 2(\ell+w)h' = 2(2+2)(2) + 2(1+2)(2) + 2(1+2)(-3) = 10\text{m}^2/\text{s}.$$

(c) The length of a diagonal is given by

$$D(\ell, w, h) = \sqrt{\ell^2 + w^2 + h^2},$$

so

$$\frac{d}{dt} D = \frac{1}{\sqrt{\ell^2 + w^2 + h^2}} \left(\ell\ell' + ww' + hh'\right) = \frac{(1)(2) + (2)(2) + (2)(-3)}{\sqrt{1^2 + 2^2 + 2^2}} = 0\text{m/s}.$$

$\square$

Problem 7.

(a) Given that f is a differentiable function with $f(2,5) = 6$, $f_x(2,5) = 1$, and $f_y(2,5) = -1$, use the linear approximation to estimate $f(2.2, 4.9)$.

(b) Generalize the formula for linear approximations to functions of three variables, find the linear approximation to the function $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ at $(3,2,6)$ and use this linear approximation to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$ (don't use the exact formula).

Solution.

(a) The linear approximation here is

$$L(x,y) = 6 + 1(x-2) - (y-5),$$

so

$$f(2.2, 4.9) \approx L(2.2, 4.9) = 6 + (0.2) - (-0.1) = 6.3.$$

(b) The generalization to three variables is

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0).$$

In this case the partial derivatives are given by

$$f_x = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad f_z = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Evaluating at $(3,2,6)$ and computing $L(x,y,z)$ gives

$$L(x,y,z) = 7 + \tfrac{3}{7}(x-3) + \tfrac{2}{7}(y-2) + \tfrac{6}{7}(z-6).$$

Then

$$f(3.02, 1.97, 5.99) \approx L(3.02, 1.97, 5.99) = 7 + \tfrac{3}{7}(0.02) + \tfrac{2}{7}(-0.03) + \tfrac{6}{7}(-0.01) = 7 - \tfrac{6}{700}.$$

$\square$