Calc III: Workshop 7 Solutions, Fall 2018

Problem 1. A lamina occupies the region in the positive quadrant (where $x \geq 0$ and $y \geq 0$) which lies inside the circle $x^2 + y^2 = 2$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

Solution. First we compute the mass of the lamina, given by

$$\begin{aligned} M &= \int_0^{\pi/2} \int_1^{\sqrt{2}} \frac{k}{r} r \, dr \, d\theta \\ &= \frac{\pi}{2} \int_1^{\sqrt{2}} k \, dr \\ &= \frac{k(\sqrt{2}-1)\pi}{2}. \end{aligned}$$

By symmetry, it follows that the center of mass $(\overline{x}, \overline{y})$ lies on the line $y = x$, and so will satisfy $\overline{x} = \overline{y}$. Thus we compute

$$\begin{aligned} \overline{y} = \overline{x} &= \frac{1}{M} \int_0^{\pi/2} \int_1^{\sqrt{2}} r \cos\theta \frac{k}{r} r \, dr \, d\theta \\ &= \frac{1}{M} \int_1^{\sqrt{2}} kr \, dr \\ &= \frac{k(2-1)}{2M} \\ &= \frac{k}{2M} \\ &= \frac{1}{(\sqrt{2}-1)\pi}. \end{aligned}$$

□

Problem 2. Find the volume of the solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$.

Solution. There are several ways to set up the integral, but it is convenient to integrate first in y:

$$\begin{aligned} \iiint_C dV &= \int_{-2}^{2} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{-1}^{4-z} dy \, dz \, dx \\ &= \int_{-2}^{2} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} 3 - z dz \, dx \end{aligned}$$

At this point in the computation, it may be convenient to switch to polar coordinates in x and z:

$$\int_{-2}^{2} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} 3 - z dz \, dx = \int_0^{2\pi} \int_0^2 (3 - r\cos\theta) \, r \, dr \, d\theta = 2\pi \tfrac{3}{2}(2)^2 = 12\pi.$$

□

Problem 3. Write five other iterated integrals that are equivalent to the iterated integral

$$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\, dz\, dx\, dy.$$

Solution. Draw the 3D picture and all three 2D projections onto coordinate planes! Then it follows that

$$\begin{aligned}
&\int_0^1 \int_y^1 \int_0^y f(x,y,z)\, dz\, dx\, dy \\
&= \int_0^1 \int_0^x \int_0^y f(x,y,z)\, dz\, dy\, dx \\
&= \int_0^1 \int_z^1 \int_z^x f(x,y,z)\, dy\, dx\, dz \\
&= \int_0^1 \int_0^x \int_z^x f(x,y,z)\, dy\, dz\, dx \\
&= \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\, dx\, dy\, dz \\
&= \int_0^1 \int_0^y \int_y^1 f(x,y,z)\, dx\, dz\, dy
\end{aligned}$$

□

Problem 4. Evaluate the triple integral $\iiint_C (4 + 5x^2yz^2)\, dV$ (try using only geometric interpretation and symmetry), where C is the cylindrical region $x^2 + y^2 \leq 4$, $-2 \leq z \leq 2$.

Solution. The region is symmetric with respect to reflection about the coordinate planes, and in particular with respect to refletion about $y = 0$. In the integrand, the second term, $5x^2yz^2$, is odd with respect to reflection about $y = 0$, so this term will integrate to 0. This leaves the integral of the first term $\iiint_C 4\, dV = 4 \iiint_C dV$, which computes 4 times the volume of C. Thus

$$\iiint_C (4 + 5x^2yz^2)\, dV = 4\mathrm{Vol}(C) = 4(\pi(2)^2)(4) = 64\pi.$$

□

Problem 5. Find the center of mass of the solid E of constant density, which lies above the xy-plane and below the paraboloid $z = 1 - x^2 - y^2$. [Hint: it may be useful to switch to polar coordinates after integrating in z.]

Solution. We may as well assume the constant in the density is 1, since its precise value is canceled when computing the center of mass.
We can write E as the region $0 \leq z \leq 1 - x^2 - y^2$, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$, $-1 \leq x \leq 1$, so that the integral giving its mass is

$$M = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} 1\, dz\, dy\, dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 - x^2 - y^2\, dy\, dx$$

At this point, we note it is convenient to switch the remaining double integral over to polar coordinates:

$$
\begin{aligned}
M &= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta \\
&= 2\pi\left(\tfrac{1}{2} - \tfrac{1}{4}\right) \\
&= \frac{\pi}{2}.
\end{aligned}
$$

By symmetry, the x and y coordinates of the center of mass must vanish, so we need only compute

$$
\begin{aligned}
\overline{z} &= \frac{1}{M} \iiint_E z\, dV \\
&= \frac{1}{M} \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} z\, r\, dr\, d\theta \\
&= \frac{2\pi}{M} \int_0^1 \tfrac{1}{2}(1 - r^2)^2 r\, dr \\
&= -\frac{\pi}{2M} \frac{(1 - r^2)^2}{2}\Big|_{r=0}^{1} \\
&= \frac{\pi}{4M} = \frac{1}{2}.
\end{aligned}
$$

□

Problem 6 (Optional bonus)**.** The *simplex of dimension* n is the set T_n of points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ bounded by the "hyperplanes" $x_i = 0$ for $i = 1, \ldots, n$ and $x_1 + x_2 + \cdots + x_n = 1$. For $n = 2$ this is a triangle and for $n = 3$ it is a tetrahedron. Compute the 4-dimensional volume of the 4-simplex:

$$
\mathrm{Vol}(T_4) = \int\!\!\int\!\!\int\!\!\int 1\, dx_4\, dx_3\, dx_2\, dx_1
$$

Formulate a conjecture for the n-dimensional volume of the n-simplex, and try and prove it!

Solution. The 4-volume is

$$\begin{aligned}
\mathrm{Vol} &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} \int_0^{1-x_1-x_2-x_3} dx_4\, dx_3\, dx_2\, dx_1 \\
&= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} (1-x_1-x_2-x_3)\, dx_3\, dx_2\, dx_1 \\
&= -\frac{1}{2} \int_0^1 \int_0^{1-x_1} (1-x_1-x_2-x_3)^2 \Big|_{x_3=0}^{1-x_1-x_2} dx_2\, dx_1 \\
&= \frac{1}{2} \int_0^1 \int_0^{1-x_1} (1-x_1-x_2)^2\, dx_2\, dx_1 \\
&= -\frac{1}{6} \int_0^1 (1-x_1-x_2)^3 \Big|_{x_2=0}^{1-x_1} dx_1 \\
&= \frac{1}{6} \int_0^1 (1-x_1)^3\, dx_1 \\
&= -\frac{1}{24} (1-x_1)^4 \Big|_{x_1=0}^{1} \\
&= \frac{1}{24}.
\end{aligned}$$

Note that it helps immensely to do each integral as a u-substitution where $u = 1 - x_1 - x_2 - \cdots - x_n$.

The area of the 2-simplex is $\frac{1}{2}$, the volume of the 3-simplex is $\frac{1}{6}$, and in general the n-volume of the n-simplex is $\frac{1}{n!}$, which can be computed as above for general n. $\square$