Calc III: Workshop 8 Solutions, Fall 2018

Problem 1. Find the center of mass of the solid S bounded by the parabolid $z = 4x^2 + 4y^2$ and the plane $z = a$ (where $a > 0$) if S has constant density.

Solution. The constant density will cancel out of the center of mass computation, so we may as well assume that the density is 1, so mass equals volume. By symmetry, the center of mass will have coordinates $(0, 0, \overline{z})$, where

$$\overline{z} = \frac{\iiint_S z\,dV}{\iiint_S dV}.$$

It is convenient to set $a = b^2$ for some b and substitute back at the end. Then S is parameterized by $4r^2 \le z \le b^2$, $0 \le r \le b/2$ (solve $z = 4r^2 = b^2$ for r), and $0 \le \theta \le 2\pi$. The denominator is given by

$$\begin{aligned}
\iiint_S dV &= \int_0^{2\pi} \int_0^{b/2} \int_{4r^2}^{b^2} r\,dz\,dr\,d\theta \\
&= 2\pi \int_0^{b/2} r(b^2 - 4r^2)\,dr \\
&= 2\pi \left(\tfrac{b^2}{2} r^2 - r^4\right)\Big|_{r=0}^{b/2} \\
&= 2\pi \left(\tfrac{b^4}{2^3} - \tfrac{b^4}{2^4}\right) \\
&= \frac{b^4\pi}{8}.
\end{aligned}$$

Then the numerator is given by

$$\begin{aligned}
\iiint_S z\,dV &= \int_0^{2\pi} \int_0^{b/2} \int_{4r^2}^{b^2} zr\,dz\,dr\,d\theta \\
&= 2\pi \tfrac{1}{2} \int_0^{b/2} (b^4 - 16r^4) r\,dr \\
&= \pi \left(\tfrac{b^4}{2} r^2 - \tfrac{16}{6} r^6\right)\Big|_{r=0}^{b/2} \\
&= \frac{b^6\pi}{12}
\end{aligned}$$

Evaluating the quotient gives

$$\overline{z} = \frac{2b^2}{3} = \frac{2a}{3}.$$

□

Problem 2. Evaluate $\iiint_E x\,dV$, where E is enclosed by the planes $z = 0$ and $z = x + y + 5$ and the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Solution. Because the region lies in between two concentric cylinders, it is most convenient to use cylindrical coordinates for this problem. The equations for the bounding surfaces in cylindrical coordinates become $z = 0$, $z = r(\cos\theta + \sin\theta) + 5$, $r = 2$ and $r = 3$. The integrand $f(x, y, z) = x$ becomes $f(r, \theta, z) = r\cos\theta$. Thus the integral is

$$\begin{aligned}
\iiint_E x\,dV &= \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta+5} r\cos\theta\, r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta+5} r\cos\theta\, r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_2^3 (r\cos\theta + r\sin\theta + 5)r^2\cos\theta\,dr\,d\theta \\
&= \int_0^{2\pi}\int_2^3 r^3\cos^2\theta + r^3\cos\theta\sin\theta + 5r^2\cos\theta\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{81-16}{4}\left(\cos^2\theta + \cos\theta\sin\theta\right) + \frac{5(27-8)}{3}\cos\theta\,d\theta \\
&= \frac{65\pi}{4}.
\end{aligned}$$

□

Problem 3. Evaluate the integral

$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\,dz\,dy\,dx$$

by changing to cylindrical coordinates.

Solution. Drawing the region in the xy-plane determined by $\left\{(x, y) : 0 \le y \le \sqrt{9-x^2},\ -3 \le x \le 3\right\}$ and the surfaces $z = 0$ and $z = 9 - x^2 - y^2$, we see that the region is bounded by the planes $z = 0$ and $y = 0$ and the paraboloid $z = 9 - x^2 - y^2$. Converting to cylindrical coordinates, we can parameterize the integral by $0 \le z \le 9 - r^2$, $0 \le r \le 3$, and $0 \le \theta \le \pi$. The integrand $\sqrt{x^2+y^2}$ just becomes r, so

$$\begin{aligned}
\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\,dz\,dy\,dx &= \int_0^{\pi}\int_0^3\int_0^{9-r^2} r^2\,dz\,dr\,d\theta \\
&= \pi\int_0^3 (9-r^2)r^2\,dr \\
&= \pi\int_0^3 9r^2 - r^4\,dr \\
&= \pi\left(3(3)^3 - \tfrac{1}{5}(3)^5\right) \\
&= \frac{162\pi}{5}.
\end{aligned}$$

□

Problem 4. Evaluate the integral

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$$

by changing to spherical coordinates.

Solution. Drawing the region bounded by the cone $z = \sqrt{x^2+y^2}$ and the hemisphere $z = \sqrt{2-x^2-y^2}$ which lies over the region $\left\{(x,y) : 0 \le y \le \sqrt{1-x^2}\right\}$ in the xy-plane, we see that the solid is bounded by the sphere $x^2+y^2+z^2 = 2$, the cone $z = \sqrt{x^2+y^2}$, and the planes $z = 0$, $x = 0$ and $y = 0$. In spherical coordinates, these equations are, respectively $\rho = \sqrt{2}$, $\phi = \frac{\pi}{4}$, $\theta = 0$, and $\theta = \frac{\pi}{2}$. The integrand xy becomes $\rho^2 \sin^2\phi \cos\theta \sin\theta$, and the volume element $dz\,dy\,dx$ becomes $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. Thus we get

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx &= \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^4 \sin^3\phi\, \cos\theta\, \sin\theta\, d\rho\, d\phi\, d\theta \\
&= \int_0^{\pi/2} \cos\theta\, \sin\theta\, d\theta \int_0^{\pi/4} \sin^3\phi\, d\phi \int_0^{\sqrt{2}} \rho^4\, d\rho \\
&= \left(\frac{1}{2}\right)\left(\frac{2^{5/2}}{5}\right) \int_0^{\pi/4} \sin\phi (1-\cos^2\phi)\, d\phi \\
&= \frac{2^{3/2}}{5}\left(-\cos\phi + \tfrac{1}{3}\cos^3\phi\right)\Big|_{\phi=0}^{\pi/4} \\
&= \frac{2^{5/2}-5}{15}.
\end{aligned}$$

□

Problem 5. Use either cylindrical or spherical coordinates to find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$. [The resulting region looks like an orange slice.]

Solution. We can assume the two planes are given by $\theta = 0$ and $\theta = \pi/6$, and hence in spherical coordinates the volume is given by

$$\begin{aligned}
\int_0^{\pi/6} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta &= \left(\frac{\pi}{6}\right)(2)\left(\frac{a^3}{3}\right) \\
&= \frac{\pi a^3}{9} = \frac{1}{12}\left(\frac{4\pi a^3}{3}\right).
\end{aligned}$$

□

Problem 6. Evaluate $\iiint_E (x^2+y^2)\,dV$, where E lies in between the spheres $x^2+y^2+z^2 = 4$ and $x^2+y^2+z^2 = 9$.

Solution. Since the region lies between consecutive spheres, spherical coordinates are best. The limits will be $2 \le \rho \le 3$, $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. The integrand x^2+y^2 becomes

$\rho^2 \sin^2 \phi$. Thus

$$\begin{aligned}\iiint_E (x^2 + y^2)\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_2^3 \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} \sin^3 \phi \, d\phi \int_2^3 \rho^4 \, d\rho \\ &= 2\pi \left(2 - \frac{2}{3}\right) \left(\frac{3^5 - 2^5}{5}\right) \\ &= \frac{1688\pi}{15}.\end{aligned}$$

$\square$

Remark. An identity that may be useful is $\sin^3 \phi = \sin \phi (1 - \cos^2 \phi)$.