Calc III: Workshop 9.5 (was 10, got renumbered) Solutions, Fall 2018

Problem 1.

(a) Verify that the vector field $\mathbf{F}(x,y,z) = (2xy + ye^x)\mathbf{i} + (x^2 + e^x)\mathbf{j}$ is conservative, and find a potential function $f(x,y)$.

(b) Compute the line integral $\int_C \mathbf{F}(x,y) \cdot d\mathbf{r}$, where C is any curve from $(0,1)$ to $(1,2)$.

Solution.

(a) We test $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + e^x) = 2x + e^x = \frac{\partial}{\partial y}(2xy + ye^y) = \frac{\partial P}{\partial y}$, so $\mathbf{F}$ is conservative. A potential function f must satisfy

$$f_x = 2xy + ye^x$$
$$f_y = x^2 + e^x$$

Integrating both equations in x and y, respectively, gives

$$f(x,y) = x^2y + ye^x + c_1(y)$$
$$f(x,y) = x^2y + ye^x + c_2(x)$$

and we see that we can take $c_1(y) = c_2(x) = 0$, so $f(x,y) = x^2y + ye^x$ is a potential function.

(b) By the fundamental theorem for line integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1,2) - f(0,1) = 2e - 1.$$

$\square$

Problem 2.

(a) The vector field $\mathbf{F}(x,y,z) = \sin y\,\mathbf{i} + (x\cos y + \cos z)\,\mathbf{j} - y\sin z\,\mathbf{k}$ is conservative. Find a potential function $f(x,y,z)$.

(b) Compute the line integral $\int_C \mathbf{F}(x,y,z) \cdot d\mathbf{r}$, where C is the parameterized curve $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}$, $0 \le t \le \pi/2$.

Solution.

(a) A potential function must satisfy

$$f_x = \sin y$$
$$f_y = x\cos y + \cos z$$
$$f_z = -y\sin z$$

Integrating the three equations with respect to x, y, and z, we get

$$f(x,y,z) = x\sin y + c_1(y,z)$$
$$f(x,y,z) = x\sin y + y\cos z + c_2(x,z)$$
$$f_z = y\cos z + c_3(x,y)$$

and we see that we can take $c_1(y,z) = y\cos z$, $c_2(x,z) = 0$, and $c_3(x,y) = x\sin y$. Thus a potential function is $f(x,y,z) = x\sin y + y\cos z$.

(b) By the fundamental theorem for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(1,\pi/2,\pi) - f(0,0,0) = 1 - \frac{\pi}{2}$$

□

Problem 3. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, and R have continuous first order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Solution. If $\mathbf{F}$ is conservative, then $P = f_x$, $Q = f_y$, and $R = f_z$ for some function $f(x,y,z)$. Then

$$\frac{\partial P}{\partial y} = f_{xy} = f_{yx} = \frac{\partial Q}{\partial x} \quad \frac{\partial P}{\partial z} = f_{xz} = f_{zx} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = f_{yz} = f_{zy} = \frac{\partial R}{\partial y}.$$

□

Problem 4. Use the previous exercise to show that the line integral $\int_C (y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k})\cdot d\mathbf{r}$ is not independent of path.

Solution. Since $\frac{\partial R}{\partial x} = yz \neq 0 = \frac{\partial P}{\partial z}$, the integrand cannot be a conservative vector field, so the integral is not independent of the path. □

Problem 5. Let $\mathbf{F}(x,y) = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2+y^2}$.

(a) Show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

(b) Show that $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is not independent of path. [Hint: compute $\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$ and $\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$ where C_1 and C_2 are the upper and lower halves of the unit circle from $(1,0)$ to $(-1,0)$.] Does this contradict the theorem that says if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on a simply connected region, then $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is path independent? Why not?

Solution.

(a) We have

$$Q_x = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2}, \qquad P_y = \frac{\partial}{\partial y}\frac{-y}{x^2+y^2} = \frac{-(x^2+y^2) + y(2y)}{(x^2+y^2)^2},$$

so

$$Q_x - P_y = \frac{x^2+y^2-2x^2+(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{2(x^2+y^2)-2(x^2+y^2)}{(x^2+y^2)^2} = 0.$$

(b) C_1 is parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ where $0 \le t \le \pi$, and $\mathbf{F}\big(\mathbf{r}(t)\big) = \frac{1}{\cos^2 t + \sin^2 t}\langle -\sin t, \cos t\rangle = \langle -\sin t, \cos t\rangle$. Then

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\; dt = \int_0^\pi 1\,dt = \pi.$$

On the other hand, C_2 is parameterized by $\mathbf{r}(t) = \langle \cos t, -\sin t\rangle$ for $0 \le t \le \pi$. We have

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \langle \sin t, \cos t\rangle \cdot \langle -\sin t, -\cos t\rangle\; dt = \int_0^\pi -1\,dt = -\pi$$

so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, despite the fact that C_1 and C_2 have the same endpoints.

This is not a contradiction to the theorem, since $\mathbf{F}$ itself, and hence the derivatives of P and Q, are not defined at $(0,0)$, thus the equation $Q_x = P_y$ can only be said to hold on $\mathbb{R}^2 \setminus (0,0)$, which is not a simply connected region, so we can't conclude that $\mathbf{F}$ is conservative.

$\square$

Problem 6. Let $\mathbf{F} = \nabla f$, where $f(x,y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0, \qquad \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1.$$

Solution. There are many possible answers. It suffices to let C_1 be any curve, say, from $(0,0)$ to $(\pi, 0)$, and C_2 any curve from $(0,0)$ to $(\pi/2, 0)$. $\square$