Calc III: Workshop 9 (Exam 2 review) Solutions, Fall 2018

Problem 1. Find the center of mass $(\overline{x}, \overline{y})$ of the quarter disk

$$Q = \{(x, y) : x^2 + y^2 \leq 1,\ 0 \leq x,\ 0 \leq y\}$$

assuming Q has unit density ($\delta(x, y) = 1$).

Solution. By symmetry $\overline{x} = \overline{y}$, where $(\overline{x}, \overline{y})$ are the coordinates of the center of mass. Also, since the density is equal to 1, the total mass is just the area, which is $\frac{1}{4}\pi$, one quarter of the area of the unit disk. Thus

$$\begin{aligned}
\overline{x} &= \frac{1}{M} \iint_Q x\, dA \\
&= \frac{4}{\pi} \int_0^{\pi/2} \int_0^1 r \sin\theta\, r\, dr\, d\theta \\
&= \frac{4}{\pi} \int_0^{\pi/2} \sin\theta\, d\theta \int_0^1 r^2\, dr \\
&= \frac{4}{3\pi}
\end{aligned}$$

So the center of mass is $(4/3\pi, 4/3\pi)$. □

Problem 2. Compute the mass of the region E enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$, assuming its density is given by $\delta(x, y, z) = z$.

Solution. E is parameterized in cylindrical coordinates by $r^2 \leq z \leq 4$, $0 \leq r \leq 2$ (the upper limit is given by setting $z = x^2 + y^2 = r^2$ equal to $z = 4$), and $0 \leq \theta \leq 2\pi$. Thus its mass is given by

$$\begin{aligned}
\text{Mass}(E) = \iiint_E z\, dV &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 z\, r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 \tfrac{r}{2}(z^2)\Big|_{z=r^2}^4 dr\, d\theta \\
&= 2\pi \int_0^2 8r - \tfrac{1}{2}r^5\, dr \\
&= 2\pi(4r^2 + \tfrac{1}{12}r^6)\Big|_{r=0}^2 \\
&= \frac{64\pi}{3}.
\end{aligned}$$

□

Problem 3. For the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} f(x, y, z)\, dz\, dy\, dx$$

(a) Change the order of integration from $dz\, dy\, dx$ to $dx\, dy\, dz$, giving the new limits (Hint: for help drawing the 3D region of integration, draw the 2D region indicated by $0 \leq y \leq \sqrt{1 - x^2}$, $0 \leq x \leq 1$ along with the surfaces $z = 0$ and $z = 1 - x^2 - y^2$)

(b) Change variables to cylindrical coordinates, giving the new limits in (z, r, θ).

Solution. Drawing the region of integration, we see that it is bounded by the planes $z = 0$, $x = 0$, $y = 0$, and the paraboloid $z = 1 - x^2 - y^2$.

(a) Changing the order to $dx\,dy\,dz$, we have

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{\sqrt{1-z-y^2}} f(x, y, z)\,dx\,dy\,dz$$

(b) Changing to cylindrical coordinates, we have

$$\int_0^{\pi/2} \int_0^1 \int_0^{1-r^2} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

□

Problem 4. Evaluate $\iiint_E y\,dV$, where E is the solid hemisphere inside $x^2 + y^2 + z^2 = 9$ where $y \geq 0$.

Solution. E is parameterized in spherical coordinates by $0 \leq \rho \leq 3$, $0 \leq \theta \leq \pi$ (only half way around), and $0 \leq \phi \leq \pi$. Then

$$\begin{aligned} \iiint_E y^2\,dV &= \int_0^\pi \int_0^\pi \int_0^3 (\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^\pi \sin\theta\,d\theta \int_0^\pi \sin^2\phi\,d\phi \int_0^3 \rho^3\,d\rho \\ &= (2)(\pi/2)(\tfrac{3^4}{4}) \\ &= \frac{81\pi}{4} \end{aligned}$$

□

Problem 5. Evaluate the line integral $\int_C x\,ds$, where C is the curve along $y = x^2$ from $(0, 0)$ to $(2, 4)$.

Solution. We may parameterize the curve by $\mathbf{r}(t) = \langle t, t^2 \rangle$, where $0 \leq t \leq 2$. Then $\mathbf{r}'(t) = \langle 1, 2t \rangle$, so the arc length element is given by $ds = |\mathbf{r}'(t)|\,dt = \sqrt{4t^2 + 1}\,dt$. The line integral is

$$\begin{aligned} \int_C x\,ds &= \int_0^2 t\sqrt{4t^2+1}\,dt \\ &= \frac{1}{8}\int_1^{17} \sqrt{u}\,du \\ &= \frac{(17)^{3/2}}{12} - \frac{1}{12}. \end{aligned}$$

□

Problem 6. Let

$$\mathbf{F}(x, y, z) = (2xy + 1)z\mathbf{i} + x^2z\mathbf{j} + (x^2y + x + 2z)\mathbf{k}.$$

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$ where C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$.

Solution. We can parameterize C by $\mathbf{r}(t) = \langle 0,0,0\rangle + t\langle 1-0, 2-0, 3-0\rangle = \langle t, 2t, 3t\rangle$, where $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 1,2,3\rangle$ and

$$\mathbf{F}\big(\mathbf{r}(t)\big) = \big\langle (2(t)(2t)+1)3t,\ t^2(3t),\ (t^2(2t)+t+2(3t))\big\rangle = \big\langle 12t^3+3t,\ 3t^3,\ 2t^3+7t\big\rangle$$

so

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= \int_0^1 12t^3+3t+2(3t^3)+3(2t^3+7t)\,dt \\ &= \int_0^1 24t^3+24t\,dt \\ &= \frac{24}{4}+\frac{24}{2} = 18.\end{aligned}$$

□