Complex Analysis, Supplementary Homework Problems 1

Problem 1. Determine the maximum of $|f|$ on $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ for

$$\text{(a)} \quad f(z) = e^{z^2} \qquad \text{(b)} \quad f(z) = \frac{z+3}{z-3}$$
$$\text{(c)} \quad f(z) = 3 - |z|^2$$

Do any of these examples violate the maximum principle? Why or why not?

Problem 2. Which of the following functions have a removable singularity at $z = 0$?

$$\text{(a)} \quad \frac{e^z}{z^{17}} \qquad \text{(b)} \quad \frac{z}{e^z - 1}$$
$$\text{(c)} \quad \frac{\cos z - 1}{z^2}$$

Problem 3. Prove the complex version of L'Hopital's rule: if $f, g : \Omega \longrightarrow \mathbb{C}$ are holomorphic with zeroes at $w \in \Omega$ of the same order k, then the function $h = \frac{f}{g}$ has a removable singularity at w and

$$\lim_{z \to w} \frac{f(z)}{g(z)} = \frac{f^{(k)}(w)}{g^{(k)}(w)}.$$

Problem 4. Find the residues of the following meromorphic functions at all singular points:

$$\text{(a)} \quad \frac{z^3}{(1+z)^3}, \qquad \text{(b)} \quad \frac{1}{(z^2+1)^3}$$
$$\text{(c)} \quad \frac{1}{(z^2+1)(z-1)^2}$$

Problem 5. Suppose f is a holomorphic function having exactly one simple zero inside the closed disk $\overline{D}_R = \{|z| \leq R\}$ and not on the boundary $C_R = \partial \overline{D}_R$. Show that the location of the zero is given by

$$w = \frac{1}{2\pi i} \int_{C_R} \frac{z f'(z)}{f(z)} \, dz.$$

Problem 6. Use Rouché's theorem to determine the number of zeroes of the polynomial $z^7 - 5z^4 + iz^2 - 2$ inside the disk $\{|z| < 1\}$. (Hint: Let $f(z) = -5z^4$.)

Problem 7.

(a) For each of the following complex numbers z, calculate the principal value of the logarithm $\mathrm{Log}(z)$:

$$i, \quad -i, \quad -1, \quad x \in (0, \infty), \quad 1+i$$

(b) Calculate the principal values of the following numbers and compare them:

$$\big(i(i-1)\big)^i \quad \text{and} \quad i^i (i-1)^i.$$