18.02A Final Exam Review

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1 The Basics

1.1 Functions, Vector Fields & Derivative Operations

The basic players in vector calculus are *functions* ($f(x,y)$ or $f(x,y,z)$) and *vector fields*. A **vector field** is a function which assigns a vector to every point in the plane (2D) or in space (3D). We can consider each component as a function and write

$$\begin{aligned} \mathbf{F}(x,y) &= M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j} \qquad \text{in 2D} \\ \mathbf{F}(x,y,z) &= M(x,y,z)\,\mathbf{i} + N(x,y,z)\,\mathbf{j} + P(x,y,z)\,\mathbf{k} \qquad \text{in 3D} \end{aligned}$$

For the derivative operations, it is useful notation to think of the symbol ∇ as being a "vector" with components

$$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$$

If $f(x,y)$ or $f(x,y,z)$ is a function of 2 or 3 variables, respectively, its **gradient** is defined by

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} \qquad \text{in 2D} \\ \nabla g &= \frac{\partial g}{\partial x}\,\mathbf{i} + \frac{\partial g}{\partial y}\,\mathbf{j} + \frac{\partial g}{\partial z}\,\mathbf{k} \qquad \text{in 3D} \end{aligned}$$

The **divergence** of a vector field is defined by

$$\begin{aligned} \text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} &= \frac{\partial}{\partial x}M(x,y) + \frac{\partial}{\partial y}N(x,y) \qquad \text{in 2D} \\ \text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} &= \frac{\partial}{\partial x}M(x,y,z) + \frac{\partial}{\partial y}N(x,y,z) + \frac{\partial}{\partial z}P(x,y,z) \qquad \text{in 3D} \end{aligned}$$

The divergence $\nabla\cdot\mathbf{F}$ at (x,y,z) measures the "source rate" of $\mathbf{F}$ at (x,y,z), or infinitesimal flux of $\mathbf{F}$ out of the point (x,y,z) (similarly in 2 dimensions). It is related to flux through the Divergence Theorem and Green's Theorem (flux form), as discussed in section 3.3.

The **curl** of a vector field in 3D is the vector field we get by computing $\nabla\times\mathbf{F}$:

$$\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} = (P_y - N_z)\,\mathbf{i} + (M_z - P_x)\,\mathbf{j} + (N_x - M_y)\,\mathbf{k}$$

If we use the above formula for a 2D vector field we get

$$\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M(x,y) & N(x,y) & 0 \end{vmatrix} = (N_x - M_y)\,\mathbf{k} \qquad \text{(since } M \text{ and } N \text{ don't depend on } z\text{)}$$

Therefore, we can consider the 2D curl of $\mathbf{F}$ to be just a function, rather than a vector field, given by

$$\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = (N_x - M_y)$$

The curl of a vector field at (x,y,z) measures the infinitesimal rate of the "twisting" of the vector field at the point (x,y,z), and it is related to work integrals through Stokes' Theorem and Green's Theorem (work form), as in section 2.3.

1.2 Simply Connected Regions

In the Fundamental Theorem of Calculus for Line Integrals, it is important that certain regions in 2D or 3D be *simply connected*. The technical definition is as follows:

Definition. *A region R is* **simply connected** *if every closed curve C, lying entirely within R, can be continuously contracted to a point, without any part of C ever leaving R.*

Intuitively, R is simply connected if it has no "holes," but you have to be careful since the meaning of a "hole" is slightly different in 2D versus 3D.

Example. *Which of the following regions are simply connected?*

- *The plane minus a point*
- *The plane minus a line*
- *3-space minus a point*
- *3-space minus a line*

Solution. The plane minus a point is *not* simply connected, since a curve encircling the removed point cannot be fully contracted without passing through the deleted region. However, the plane minus a line *is* simply connected, since any closed curve must start out being entirely in one half or the other (otherwise it would not be "entirely contained in R"), and then in either case can clearly be contracted to a point.

In 3-space, the situation is reversed. If we delete a point and consider a curve encircling the point, we can move the curve "sideways," away from the deleted point before we contract it. However, a curve which encircles the deleted line cannot be fully contracted without passing through the line. $\square$

1.3 Line Integrals

A line integral over a curve C in 2 or 3 dimensions an expression of the form

$$\int_C M(x,y)\,dx + N(x,y)\,dy \qquad \text{in 2D}$$

$$\int_C M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz \qquad \text{in 3D}$$

The quantities $(M\,dx+N\,dy)$ and $(M\,dx+N\,dy+P\,dz)$ are called *differentials*. **To calculate such integrals, we must *parameterize* the curve** C by letting x, y (and z) be functions of an independent variable, say t. When it is convenient, we sometimes let the independent variable be one of x, y, or z, writing the other two as functions of it.

Example. *Calculate*

$$\int_C -zy\,dx + zx\,dy + z\,dz$$

Where C is the helix $x = \cos t$, $y = \sin t$, $z = t$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

Solution. The parameterization of C is as above. In this parameterization we have $z = t$, so we could choose to let z be the independent variable if we like. Then $x = \cos z$, $y = \sin z$, $z = z$. We let z run from 0 to 2π to trace out the curve. We have

$$dx = -\sin z\,dz \quad dy = \cos z\,dz \quad dz = dz$$

and, substituting in for x and y, we get

$$\int_C -zy\,dx + zx\,dy + z\,dz = \int_0^{2\pi} z\sin^2 z\,dz + z\cos^2 z\,dz + z\,dz = \int_0^{2\pi} 2z\,dz = (2\pi)^2$$

$\square$

1.4 Volume Integrals

Volume integrals in 3D are just iterated triple integrals. The quantities we usually are interested in computing are

$$\begin{aligned}
\text{Volume} &= \iiint_V dV \\
\text{Mass} &= \iiint_V \delta\, dV \qquad \delta = \text{density} \\
\overline{x} &= \iiint_V x\delta\, dV \\
\overline{y} &= \iiint_V y\delta\, dV \\
\overline{z} &= \iiint_V z\delta\, dV \\
\text{Center of Mass} &= \frac{1}{\text{Mass}}\left(\overline{x}, \overline{y}, \overline{z}\right) \\
\text{Moment of Inertia about } z\text{-axis } I_z &= \iiint_V r^2\delta\, dV \qquad r = \text{distance to } z\text{-axis}
\end{aligned}$$

Volume integrals are calculated in one of three coordinate systems.

1.4.1 Spherical Coordinates

If the region of integration is part of a sphere (this includes spheres, hemispheres, regions shaped like "ice cream cones," "orange slices," etc.), we want to use spherical coordinates, given by[1]

$$\begin{aligned}
&\rho = \sqrt{x^2 + y^2 + z^2} \\
&z = \rho\cos\phi \\
&r = \rho\sin\phi \\
&y = r\sin\theta = \rho\sin\phi\sin\theta \\
&x = r\cos\theta = \rho\sin\phi\cos\theta \\
&dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = (d\rho)(\rho\, d\phi)(\rho\sin\phi\, d\theta)
\end{aligned}$$

See figure 1.

Example. *Compute the center of mass of a hemisphere of radius a with density $\delta = \rho$.*

Solution. To compute the center of mass, we need $\overline{x}$, $\overline{y}$, $\overline{z}$ and the mass. However, if we think of the hemisphere as sitting over the x-y plane centered about the origin, we know that $\overline{x} = \overline{y} = 0$ by symmetry (because the region of integration *and* the density are symmetric with respect to x and y).

First we compute the mass

$$\text{Mass} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = (2\pi)(1)\left(\frac{a^4}{4}\right) = \frac{\pi a^4}{2}$$

Now compute $\overline{z}$, using $z = \rho\cos\phi$:

$$\overline{z} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos\phi)(\rho)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = (2\pi)\left(\frac{1}{2}\right)\left(\frac{a^5}{5}\right) = \frac{\pi a^5}{5}$$

[1] In the last line, we have split up the factors of ρ and $\sin\phi$ among the differentials; this will be useful when computing surface integrals in spherical coordinates.

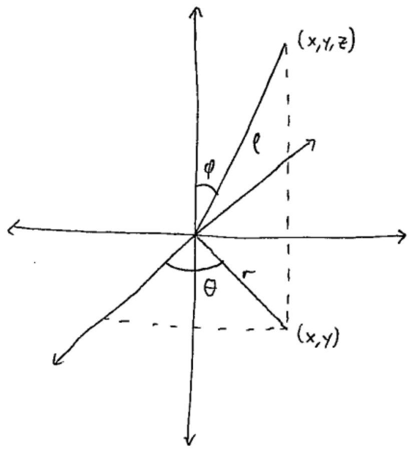

Figure 1: Spherical coordinates

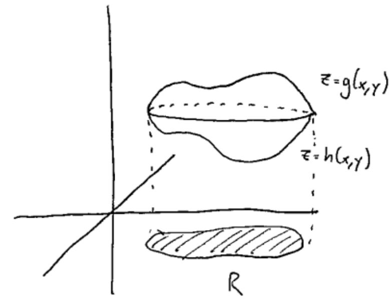

Figure 2: Volume integrals in cylindrical or cartesian coordinates.

We conclude that the center of mass is

$$\text{CM} = \frac{1}{\text{Mass}}\,(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{2a}{5}\right)$$

□

1.4.2 Cylindrical and Cartesian Coordinates

These are both distinguished by having z as a coordinate. The idea here is to do the integral in z first, which then leaves a double integral in either (x, y) (cartesian coordinates)or (r, θ) (cylindrical coordinates). If the volume V to be integrated over is bounded above by a surface $z = g(x, y)$ (or $z = g(r, \theta)$) and below by a surface $z = h(x, y)$ (or $z = h(r, \theta)$), we have

$$\iiint_V f\,dV = \iint_R \int_h^g f\,dz\,dA$$

where R is the "shadow region" described by the projection of V onto the x-y plane. See figure 2. R can be determined by setting $g(x, y) = h(x, y)$, which gives a curve describing the boundary of R.

Example. *Find the volume of the region bounded above by the paraboloid $z + x^2 + y^2 = 2$ and below by the cone $z = r$.*

Solution. We must choose whether to use cylindrical or cartesian coordinates. The shadow region will be circular, so it is best to use cylindrical coordinates. The upper surface is $z = 2 - r^2$ and the lower surface is $z = r$. To determine R, we need to determine the intersection of the upper and lower surfaces, so we solve

$$2 - r^2 = r \Longrightarrow r^2 + r - 2 = 0 \Longrightarrow (r + 2)(r - 1) = 0$$

Since r is always positive, the solution is the positive root $r = 1$. Thus R is the unit disk in the x-y plane. Now the integral is

$$\text{Volume} = \iiint_V dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} dz\,r\,dr\,d\theta = 2\pi \int_0^1 (2 - r^2 - r)r\,dr = (2\pi)\left(1 - \frac{1}{4} - \frac{1}{3}\right) = \frac{5\pi}{6}$$

□

1.5 Surface Integrals

In 2D, surface integrals (or area integrals) are just ordinary double integrals. The integral of a function $f(x, y)$ over a region R is just

$$\iint_R f\,dA = \iint_R f(x,y)\,dx\,dy \qquad \text{or} \qquad \iint_R f\,dA = \iint_R f(r,\theta)\,r\,dr\,d\theta$$

in either cartesian or polar coordinates. Remember that the area element dA in polar coordinates is $dA = r\,dr\,d\theta$.

A surface integral in 3D is more complicated. As was the case for line integrals, to evaluate a surface integral of the form

$$I = \iint_S f(x,y,z)\,dS$$

we must first *parametrize* S by two variables in order to convert I to an ordinary double integral. Typically the two parameters will be either x and y, in the case of general surfaces lying "over" the x-y plane, or two of the three spherical/cylindrical coordinates, in the case of surfaces described by setting the third variable equal to a constant. Let's deal with the latter case first.

1.5.1 Surface Integrals in Spherical/Cylindrical Coordinates

For surfaces which are easily described in terms of cartesian, cylindrical or spherical coordinates, we can derive dS from dV as follows:

1. Write down dV as a product of differentials, placing the appropriate factors with their differentials:

$$\begin{aligned} dV &= (dx)(dy)(dz) \qquad \text{for cartesian coordinates} \\ dV &= (dz)(dr)(rd\theta) \qquad \text{for cylindrical coordinates} \\ dV &= (d\rho)(\rho d\phi)(\rho\sin\phi d\theta) \qquad \text{for spherical coordinates} \end{aligned}$$

2. If the surface is described by one of the coordinates being constant, remove the corresponding differential factor from dV to get dS, for example,

$$\begin{aligned} dS &= (dz)(rd\theta) = r\,dz\,d\theta \qquad \text{for surfaces } r = \text{const} \\ dS &= (\rho d\phi)(\rho\sin\phi d\theta) = \rho^2\sin\phi\,d\phi\,d\theta \qquad \text{for surfaces } \rho = \text{const} \\ dS &= (d\rho)(\rho\sin\phi d\theta) = \rho\sin\phi\,d\rho\,d\theta \qquad \text{for surfaces } \phi = \text{const} \end{aligned}$$

Example. *Calculate the surface area of a cone with height h and cone angle ϕ_0*

Solution. Visualize standing the cone on its head along the z-axis, with its tip at the origin (see figure 3). Let $S = C + B$, where B is the base and C is the rest of the cone. Note that we can describe C by setting $\phi = \phi_0/2$, so we can compute this in spherical coordinates.

By examining the right triangle made by the z-axis, and the radius of the base, we can determine that the radius of the base is given by

$$r_0 = h\tan\phi_0/2$$

and the length ρ_0 from the tip to the outside of the base is

$$\rho_0 = \left(h^2 + r_0^2\right)^{1/2} = h\left(1 + \tan^2\phi_0/2\right)^{1/2}$$

So the area of the base is

$$\text{Area}\,(B) = \pi h^2\tan^2\phi_0/2$$

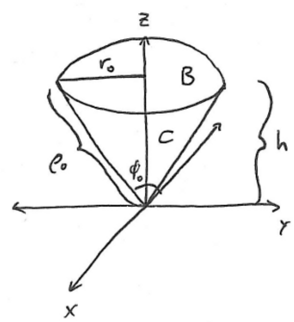

Figure 3: Computing the surface area of a cone.

and, using $dV(d\rho)(\rho d\phi)(\rho \sin\phi d\theta)$ and deleting the $d\phi$ term, the area of C is

$$\text{Area}\,(C) = \iint_C 1\, dS = \int_0^{2\pi} \int_0^{\rho_0} \rho \sin\phi_0/2\, d\rho\, d\theta = \pi\rho_0^2 \sin\phi_0/2 = \pi\rho_0 r_0$$

Using $\rho_0 \sin\phi_0/2 = r_0$ in the last equality. Thus the total area is

$$\text{Area} = \pi r_0^2 + \pi r_0 \rho_0 \qquad \rho_0 = h\left(1 + \tan^2\phi_0/2\right)^{1/2}, \quad r_0 = h\tan\phi_0/2$$

□

1.5.2 Surface Integrals over Arbitrary Surfaces

For an arbitrary surface lying over the x-y plane, we can use x and y as parameters, but we need a formula giving the area element dS in terms of $dA = dx\, dy$. **Note:** for flux integrals, it is usually easier to use the formulas that give $\mathbf{n}\, dS$ directly as in section 1.5.3. There are two ways of describing the surface S. If $z = z(x, y)$ as a function of x and y, we have

$$dS = \sqrt{z_x^2 + z_y^2 + 1}\, dx\, dy \qquad \text{if } z = z(x, y) \text{ on } S$$

If S is described as the zero set of a function $f(x, y, z) = 0$, we have

$$dS = \left|\frac{\nabla f}{f_z}\right| dx\, dy \qquad \text{if } f(x, y, z) = 0 \text{ on } S$$

Note that the first formula follows from the second; if $z = g(x, y)$, we can let $f(x, y, z) = z - g(x, y)$, whence $\nabla f \cdot \mathbf{k} = 1$ and $|\nabla f| = \sqrt{g_x^2 + g_y^2 + 1}$.

Note also that both formulas follow from the formulas for $\mathbf{n}\, dS$ below in section 1.5.3, by setting

$$dS = |\mathbf{n}\, dS|$$

So it is most useful to memorize those formulas, rather than these.

Example. *Calculate the rotational moment of inertia about the z-axis of an infinitely thin shell described by $z = x^2 + y^2$ between $z = 0$ and $z = 1$, with uniform (planar) density $\delta = 1$.*

Solution. Since the shell is very thin, we use a surface integral rather than a volumne integral, to calculate

$$I_z = \iint_S r^2\, dS$$

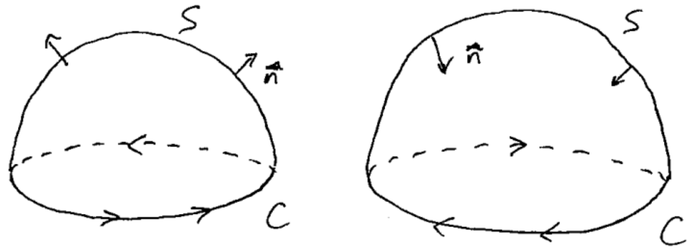

Figure 4: The relation between the orientation of S (direction of $\mathbf{n}$) and the orientation of $C = \partial S$

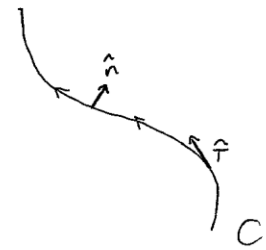

Figure 5: A curve with tangent vector $\mathbf{T}$ and normal vector $\mathbf{n}$

Using $z = x^2 + y^2$, we obtain the area element

$$dS = \sqrt{z_x^2 + z_y^2 + 1}\, dx\, dy = \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy = \sqrt{4r^2 + 1}\, r\, dr\, d\theta$$

The shadow region R is the unit circle, so

$$I_z = \int_0^{2\pi} \int_0^1 \frac{r^3}{\sqrt{r^2 + 1}}\, dr\, d\theta$$

This is a tricky integral actually (whoops!), and I'll leave it to you to check (trig substitutions!)

$$I_z = 2\pi \left(\frac{(1 + r^2)^{3/2}}{3} - (1 + r^2)^{1/2} \right) \Big|_{r=0}^{1} = 2\pi \left(\frac{2 - \sqrt{2}}{3} \right)$$

□

1.5.3 Calculating $\mathbf{n}\, dS$ for an arbitrary surface

The most common type of surface integral occurs when the integrand is not just a scalar function, but the dot product of a vector field $\mathbf{F}$ with the normal $\mathbf{n}$ to the surface S; that is,

$$I = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

We need this for the calculation of flux integrals in 3D for instance, as in section 3.2, and when invoking Stokes' Theorem as in section 2.3, in which case $\mathbf{F} = \nabla \times \mathbf{G}$ for some field G.

For an arbitrary surface lying over the x-y plane, it is easiest to calculate $\mathbf{n}\, dS$ in one shot, rather than determining $\mathbf{n}$ and dS separately. If the surface is given by z as a function of x and y, that is $z = z(x, y)$, we have

$$\mathbf{n}\, dS = (-z_x\, \mathbf{i} - z_y\, \mathbf{j} + \mathbf{k})\ dx\, dy \qquad \text{for } \mathbf{n} \text{ pointing "up"}$$

and taking the opposite vector $(z_x\, \mathbf{i} + z_y\, \mathbf{j} - \mathbf{k})\ dx\, dy$ to get the opposite orientation, with $\mathbf{n}$ pointing down (see figure 4).

If the surface is given in the form $f(x, y, z) = \text{const}$, we can use the formula

$$\mathbf{n}\, dS = \frac{\nabla f}{f_z}\, dx\, dy$$

This is the better of the two to memorize since it is more general and, as in the previous section, the first formula follows from the second. In the case $z = g(x, y)$ above, we can rewrite it in the form $f(x, y, z) = z - g(x, y) = 0$ and use the gradient formula, which will give the same answer.

Once we have parameterized the surface in terms of x and y, we have an ordinary double integral over the "shadow region" R (the projection of the surface onto the x-y plane):

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot \left(\frac{\nabla f}{f_z}\right) dx\, dy$$

Example. *Calculate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$*, where* $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$*, and* S *is the surface* $z = 4 - x^2 - y^2$ *over the* x-y *plane, with upward pointing normal vector.*

Solution. Using the first method, we get

$$\mathbf{n}\, dS = (-z_x\,\mathbf{i} - z_y\,\mathbf{j} + \mathbf{k})\; dx\, dy = (2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k})\; dx\, dy$$

Alternatively, if we think of S as being described by the function $f(x, y, z) = x^2 + y^2 + z = 4$, we obtain

$$\mathbf{n}\, dS = \frac{\nabla f}{f_z}\, dx\, dy = \frac{(2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k})}{1}\, dx\, dy$$

which of course must give the same formula. In any case, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k})\, dx\, dy = 2\iint_R (x^2 + y^2)\, dx\, dy$$

where R is the region bounded by $x^2 + y^2 = 4$, e.g. the disk of radius 2. Because both the integrand and region of integration are naturally expressed in polar coordinates, we can change coordinates to compute the integral.

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^2 (r^2) r\, dr\, d\theta = 16\pi$$

□

2 Work & Related Theorems

2.1 Work Integrals in 2D and 3D

One of the most important examples of line integrals comes from computing the *work* done by a force field over a path C. In this situation, a vector field $\mathbf{F}$ represents force throughout space, and we are interested in the work done by the force on an object which moves over the curve C. If $\mathbf{T}$ is the unit tangent vector to the curve (see figure 5) and ds is the element of arclength, the total work W is given by[2]

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot\, d\mathbf{r}$$

Suppose $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ in 2D or $\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}$ in 3D. Since $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ ($d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ in 2D), we have

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C M(x, y)\, dx + N(x, y)\, dy \qquad \text{in 2D}$$
$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C M(x, y, z)\, dx + N(x, y, z)\, dy + P(x, y, z)\, dz \qquad \text{in 3D}$$

Example. *Compute the work done by the force* $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + z\,\mathbf{k}$ *over the path* C *given by* $x = t$, $y = t^2$, $z = t^3$ *from* $(0, 0, 0)$ *to* $(1, 1, 1)$.

[2] The second equality follows from the formula $d\mathbf{r}/ds = \mathbf{T}$.

Solution. We write the work in differential form first:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C xy\,dx + yz\,dy + z\,dz$$

Then, parameterizing the curve in terms of the independent variable t, we have

$$W = \int_0^1 (t)(t^2)(dt) + (t^2)(t^3)(2t\,dt) + (t^3)(3t^2\,dt) = \int_0^1 t^3 + 2t^6 + 3t^5\,dt = \frac{1}{4} + \frac{2}{7} + \frac{1}{2} = \frac{29}{28}$$

$\square$

An important case for work integrals is when the vector field is *conservative*, which we will discuss next in section 2.2. Also Green's Theorem (work form) and Stokes' Theorem can facilitate the calculation of work integrals (see section 2.3).

2.2 Conservative Fields and the Fundamental Theorem

An important case for work integrals is when $\mathbf{F}$ is *conservative*, that is, when $\mathbf{F} = \nabla f$ for some function f. We can express this in terms of differentials by $(M\,dx + N\,dy) = df$ in 2D, or $(M\,dx + N\,dy + P\,dz) = df$ in 3D; we say such differentials are *exact*. In this case, the Fundamental Theorem of Calculus for Line Integrals applies:

Theorem 1. (Fundamental Theorem of Calculus for Line Integrals) *For any curve C with starting at p_0 and ending at p_1, we have*

$$\int_C \nabla f \cdot d\mathbf{r} = f(p_1) - f(p_0)$$

where $p_i = (x_i, y_i)$ in 2D and $p_i = (x_i, y_i, z_i)$ in 3D.

Thus, when $\mathbf{F}$ is conservative, we have the following situation:

- $\mathbf{F} = \nabla f$.
- $\int_C \mathbf{F} \cdot d\mathbf{r} = f(p_1) - f(p_0)$ for any curve C connecting p_0 to p_1; we say the integral is *path independent*.
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all *closed* paths C (since $p_0 = p_1$ for such paths).

Remark. (Optional material) The FTCLI fits in to the general framework of Green's Theorem, Stokes' Theorem and the divergence theorem. In the latter cases, we have some statement like

$$\oint_{\partial Q} (\ldots)\,d(\text{something}) = \iint_Q D(\ldots)\,d(\text{something else})$$

where the left hand side is the integral of some quantity (work, or flux, say) over the 1 (or 2) dimensional boundary ∂Q of the 2 (or 3) dimensional region Q; and D is some kind of derivative operation (like curl or divergence) applied to the integrand of the left hand side.

If we consider the *boundary* of the curve C to be the two points p_0 and p_1, with *orientation* $+1$ for p_1 and -1 for p_0, and we define the integral over a point to be just[3] $\int_p f = f(p)$, then we can write the FTCLI suggestively as

$$\oint_{\partial C} f = \oint_{\{+p_1, -p_0\}} f = f(p_1) - f(p_0) = \int_C \nabla f\,ds$$

where the derivative operation is just the gradient! Nice, huh? $\square$

[3]Indeed, what else could it be?

2.2.1 Test for F to be Conservative

We need a criterion to tell when $\mathbf{F}$ is a conservative vector field, so that in these cases, we can find its potential function f and use the FTCLI to make the evaluation of work integrals much easier. Two conditions need to be satisfied.

1. $\nabla \times \mathbf{F} = 0$ in a region[4] D
2. D is simply connected (see section 1.2)

Only if both of these conditions are satisfied can we write $\mathbf{F} = \nabla f$ for some f. To summarize

$$\nabla \times \mathbf{F} = 0 \text{ in } D;\ D \text{ simply connected} \implies \mathbf{F} = \nabla f$$

2.2.2 Recovering f from F

Given that we have a conservative vector field over a simply connected domain, we can find the function f such that $\nabla f = \mathbf{F}$ in one of two ways.

1. Set $f(x,y,z) = \int_C \mathbf{F}\cdot d\mathbf{r}$ for any curve C ending at (x,y,z). From the FTC, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = f(x,y,z) - f(\text{starting point of C})$, and since the latter term is a constant, we can disregard it[5].
2. Iteratively solve for f by making its partial derivatives with respect to x, y, and z match the components of $\mathbf{F}$. This is best illustrated by the example below.

Example. *The vector field* $\mathbf{F} = (yz+1)\,\mathbf{i} + xz\,\mathbf{j} + (xy+1)\,\mathbf{k}$ *is conservative. Find* f *by method 2.*

Solution. We first require that $f_x = M$:

$$\frac{\partial f}{\partial x} = yz + 1 \quad \implies \quad f = xyz + x + g(y,z)$$

where $g(y,z)$ is a yet-to-be-determined function of y and z. Taking the y derivative of this and comparing it to N, we have

$$\frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz \quad \implies \quad g_y = 0 \quad \implies \quad g(y,z) = h(z)$$

where h is an arbitrary function of z. Plugging this in, taking the z derivative and comparing to P, we have

$$\frac{\partial f}{\partial z} = xy + \frac{dh}{dz} = xy + 1 \quad \implies \quad h(z) = z + c$$

We don't care about constants, so we can set $c = 0$. Thus we conclude

$$f(x,y,z) = xyz + x + z$$

□

Example. *For what value of* a *is the 2D vector field* $\mathbf{F} = (axy + y^3)\,\mathbf{i} + (2x^2 + 3xy^2)\,\mathbf{j}$ *conservative? For such* a*, find* f *by method 1 above.*

Solution. To find the value of a, we compute the curl:

$$\nabla \times \mathbf{F} = \frac{\partial}{\partial x}(2x^2 + 3xy^2) - \frac{\partial}{\partial y}(axy + y^3) = 4x + 3y^2 - ax - 3y^2$$

so $\mathbf{F}$ will be conservative for $a = 4$. To obtain f, fix a point (x_0, y_0) and let C' be the curve along the x-axis, from $(0,0)$ to $(x_0, 0)$, and let C'' be the vertical curve from $(x_0, 0)$ to (x_0, y_0). Using the fact that $dy = 0$ along C' and $dx = 0$ along C'' we compute the integral

$$\int_{C'+C''} \mathbf{F}\cdot d\mathbf{r} = \int_0^{x_0} (4x(0) + (0)^3)\,dx + \int_0^{y_0} (2x_0^2 + 3x_0y^2)\,dy = 0 + (2x_0^2y_0 + x_0y_0^3) = f(x_0, y_0)$$

Letting (x_0, y_0) vary, we obtain $f(x,y) = 2x^2y + xy^3$, and indeed $\nabla f = \mathbf{F}$ as is easily verified. □

[4] D may be either a 2 or 3 dimensional region.

[5] If $g(x,y,z) = f(x,y,z) + c$, then $\nabla g = \mathbf{F}$ also, so we don't care about adding or subtracting constants.

2.3 Work: Green's Theorem and Stokes' Theorem

In addition to the Fundamental Theorem for Line Integrals, the other important theorems involving work integrals are Green's Theorem and Stokes' Theorem (really the same theorem), which relate work done over a *closed* curve, to the *curl*, integrated over a surface. Remember, **work is related to curl**.

Theorem 2. (Green's Theorem in Work Form) *Let* $\mathbf{F}$ *be a 2D vector field and* C *a simple, closed curve. If* C *is the boundary of a region* R*, (so we write* $C = \partial R$*), then*

$$\oint_{C=\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F}\, dA$$

Theorem 3. (Stokes' Theorem) *Let* $\mathbf{F}$ *be a 3D vector field and* C *a simple, closed curve. If* S *is any*[6] *surface with unit normal* $\mathbf{n}$ *that has* C *as its boundary (we write* $\partial S = C$*), then*

$$\oint_{C=\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

Remark. It is important that S and C be *compatibly oriented*. **The convention is given by the right-hand rule**: if you point your thumb in the direction of traversal of C, your fingers will curl around and cross the surface in the direction of $\mathbf{n}$. See figure 4. In Green's Theorem, the orientation convention is that R should be to the *left* as C is traversed. See figure 6.

Remark. (Optional material) Green's Theorem in Work Form is just a special case of Stokes' Theorem. Remember that if we consider $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ to be a 3 dimensional vector field independent of z and with no $\mathbf{j}$ component, we can compute the curl as in section 1.1 to be

$$\nabla \times \mathbf{F} = (N_x - M_y)\,\mathbf{k}$$

Additionally, if C is lying in the x-y plane, we can choose surface S in Stokes' Theorem to *be* the region R in the x-y plane. In this case, to be compatible with orientations, R will have to have normal vector $\mathbf{n} = \mathbf{k}$. Thus

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R ((N_x - M_y)\,\mathbf{k}) \cdot \mathbf{k}\, dS = \iint_R (N_x - M_y)\, dA$$

Convince yourself that the orientation conventions for Green's Theorem and for Stokes' Theorem work out to the same thing. Now you can feel free to think of them as the same theorem and only memorize the form of Stokes' Theorem! □

The practical use of these theorems is to simplify the calculation of work integrals. To calculate work over a closed curve, we use them directly:

Example. *What is the work done by the force field* $\mathbf{F} = x \sin z\,\mathbf{i} + xy^2\,\mathbf{j} + y^2 \cos x\,\mathbf{k}$ *over the unit circle in the* x-y *plane?*

Solution. This is a complicated vector field, and $\oint_C \mathbf{F} \cdot d\mathbf{r}$ would be a serious pain to calculate. Fortunately, we get a huge simplification using Stokes' Theorem. We get to choose the surface S which is bounded by C that we want to integrate over; let us choose one with a convenient normal vector. First compute the curl (see section 1.1):

$$\nabla \times \mathbf{F} = 2y \cos x\,\mathbf{i} + (x \cos z + y^2 \sin x)\,\mathbf{j} + y^2\,\mathbf{k}$$

Since $\nabla \times \mathbf{F}$ has such a simple $\mathbf{k}$ component, let's choose S to have normal vector $\mathbf{n} = \mathbf{k}$, that is, S is the unit disk in the x-y plane. Then by Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S y^2\, dS = \int_0^{2\pi} \int_0^1 (r \sin\theta)^2\, r\, dr\, d\theta = \frac{\pi}{4}$$

□

[6] Note that in 3D, such S is not unique; there are many surfaces which have boundary equal to any given closed curve. It is often useful to exploit this freedom (see the examples below).

If the curve C is not closed, we can often add a convenient curve C' to close it up, and get a relation between the work over C, the work over C', and the flux integral of $\nabla \times \mathbf{F}$ over the region bounded by $C + C'$.

Example. *Relate the work done by the field* $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$ *over the straight line from* $(a, 0)$ *to* $(0, b)$ *to the area of the triangle* R *with vertices* $(0, 0)$*,* $(a, 0)$ *and* $(0, b)$*.*

Solution. The straight line, call it C, is not a closed curve, but if we add the curves (be careful of orientations!) C' from $(0, b)$ to $(0, 0)$ and C'' from $(0, 0)$ to $(a, 0)$ along the axes, we do get a closed curve $(C + C' + C'')$ which bounds the triangle R in question. Since $\nabla \times \mathbf{F} = \frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(-y) = 2$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C'} \mathbf{F} \cdot d\mathbf{r} + \int_{C''} \mathbf{F} \cdot d\mathbf{r} = \oint_{C+C'+C''} \mathbf{F} \cdot d\mathbf{r} = \iint_R 2\, dA$$

Let us compute $\int_{C'} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C''} \mathbf{F} \cdot d\mathbf{r}$. Along C', we have $x = 0$, $y = y$ from b to 0, so

$$\int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} -y\, dx + x\, dy = \int_b^0 (-y)(0) + (0)\, dy = 0$$

Similarly, over C'', we have $y = 0$, $x = x$, and so

$$\int_{C''} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} -y\, dx + x\, dy = \int_0^a (0)\, dx + x(0) = 0$$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 2(\text{Area of } R)$$

□

3 Flux & Related Theorems

Another quantity of great physical importance is **flux**. If you think of a vector field as a kind of *flow*, then the flux is a measure of the flow rate *across* the region of integration. It is for this reason that flux is a *line* integral in 2D and a *surface* integral in 3D. To measure the flow *across* something, the something has to have dimension 1 less than the ambient dimension (indeed, it does not make any sense to ask what is the flow rate across a curve in 3D, or across an area in 2D). Though it is helpful to think in terms of flow, be aware that in physics, the flux does not always measure the physical movement of a quantity. For example, in electromagnetism we often consider the flux of the electric or magnetic fields over surfaces.

3.1 Flux Integrals in 2D

In any case, to calculate the flux of $\mathbf{F}$ across a curve C in 2 dimensions, we need to specify a normal vector to C. The orientation convention is to take $\mathbf{n}$ to be pointing to the *right* as we traverse C[7] (see figure 5).

If $\mathbf{T}\, ds = d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ then the correctly oriented normal is given by rotating 90° clockwise to get[8] $\mathbf{n}\, ds = dy\,\mathbf{i} - dx\,\mathbf{j}$. Then if $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$, the total flux is given by

$$\text{Flux} = \int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C M(x, y)\, dy - N(x, y)\, dx$$

[7] This is consistent with the orientation convention for Green's Theorem, in which we need the normal vector to point *away* from the region, which is to the left.

[8] To get the signs right, think of the special cases $dx = 0$ and $dy = 0$.

Example. *Calculate the flux of the vector field* $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ *across the curve* C *which is the straight line from* $(1,0)$ *to* $(0,2)$

Solution. The curve C is parametrized by $y = 2-2x$, $x = x$ with x running from 1 to 0. Since C is traversed from $(1,0)$ to $(0,2)$, the normal vector to C is $\mathbf{n} = (2\,\mathbf{i} + \mathbf{j})/\sqrt{5}$. We have

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C x\,dy - y\,dx = \int_1^0 x(-2\,dx) - (2-2x)(dx) = \int_0^1 2x + 2 - 2x\,dx = 2$$

□

3.2 Flux Integrals in 3D

Flux in 3D is similar. We have to choose a normal vector $\mathbf{n}$ to the surface S, and then

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

Using the techniques of section 1.5, we can either

1. Calculate $\mathbf{n}$ and dS separately for a given parametrization (useful when $\mathbf{n}$ has a particularly simple form, such as $\mathbf{n} = \hat{\mathbf{r}}$ and the surface is described in special coordinates by some coordinate being constant; or $\mathbf{n}$ is constant, and when $\mathbf{F}$ is everywhere parallel to $\mathbf{n}$), or
2. Calculate $\mathbf{n}\,dS$ in one shot, using the formulas in section 1.5.3

Example. *Calculate the flux of* $F = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ *through the surface* S *of the upper hemisphere of radius* a*, with normal vector pointing up.*

Solution. S is most easily described in spherical coordinates as the part of the surface $\rho = a$. Using the above, we see that $dS = \rho^2\sin\phi\,d\phi\,d\theta = a^2\sin\phi\,d\phi\,d\theta$. Since $\mathbf{n}$ and $\mathbf{F}$ are always parallel, we have

$$\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}| = \rho = a$$

Thus,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi/2}(a)a^2\sin\phi\,d\phi\,d\theta = 2\pi a^3$$

Alternatively, we could consider the parametrization in x and y (actually r and θ) as follows. Given that S is described by $f(x,y,z) = x^2+y^2+z^2-a^2 = 0$, we get

$$\mathbf{n}\,dS = \frac{\nabla f}{f_z}\,dx\,dy = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{2z}\,dx\,dy$$

Then

$$\mathbf{F}\cdot\mathbf{n}\,dS = \frac{2\left(x^2+y^2+z^2\right)}{2z}\,dx\,dy = \frac{a^2}{z}\,dx\,dy = \frac{a^2}{\left(a^2-x^2-y^2\right)^{1/2}}\,dx\,dy = \frac{r}{\left(a^2-r^2\right)^{1/2}}\,dr\,d\theta$$

We integrate over the shadow region, which is the disk of radius a, so

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = a^2\int_0^{2\pi}\int_0^a \frac{r}{\left(a^2-r^2\right)^{1/2}}\,dr\,d\theta = -2\pi a^2\left(a^2-r^2\right)^{1/2}\Big|_{r=0}^{a} = 2\pi a^3$$

□

3.3 Flux: Green's Theorem and the Divergence Theorem

For flux integrals, the two important theorems are Green's Theorem in flux form in 2D and the Divergence Theorem in 3D. Green's Theorem relates the flux integral of a vector field $\mathbf{F}$ over a *closed curve* to the integral of the divergence $\nabla \cdot \mathbf{F}$ over the region bounded by the curve. The Divergence Theorem relates the flux integral of $\mathbf{F}$ over a *closed surface*[9] to the integral of $\nabla \cdot \mathbf{F}$ over the *volume* bounded by the surface. Remember, **flux is related to divergence**.

Theorem 4. (Green's Theorem in Flux Form) *Let* $\mathbf{F}$ *be a 2D vector field and* C *a simple, closed curve with outward pointing normal vector* $\mathbf{n}$*. If* R *is a region that has* C *as its boundary (we write* $\partial R = C$*), then*

$$\oint_{C=\partial R} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

Theorem 5. (Divergence Theorem) *Let* $\mathbf{F}$ *be a 3D vector field and* S *a closed surface with outward pointing normal vector* $\mathbf{n}$*. If* D *is a region that has* S *as its boundary (we write* $\partial D = S$*), then*

$$\oiint_{S=\partial D} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

Remark. It is important that C and R or S and D are *compatibly oriented*. **The convention is that $\mathbf{n}$ should always point away from the region** (either R or D) over which we're integrating. See figure 6.

The practical value of these theorems is to facilitate the calculation of flux integrals. Over closed curves/surfaces, we can use the theorems directly.

Example. *Calculate the flux of* $\mathbf{F} = 8xy\,\mathbf{i} + (2x^2y - 4y^2)\,\mathbf{j} + (y - 2x^2z)\,\mathbf{k}$ *across the surface of the unit sphere* S*.*

Solution. Since the surface is closed, we apply the divergence theorem. We calculate

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(8xy) + \frac{\partial}{\partial y}(2x^2y - 4y^2) + \frac{\partial}{\partial z}(y - 2x^2z) = (8y) + (2x^2 - 8y) + (-2x^2) = 0$$

Thus,

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV = 0$$

□

Of course, if the curve/surface is *not* closed, it may be convenient to add additional curves/surfaces to close it up, as in the following examples.

Example. *Calculate the flux of* $\mathbf{F} = (x + yz^2)\,\mathbf{i} + (x^2z)\,\mathbf{j} + z\,\mathbf{k}$ *through* S*, where* S *is the surface of the upper hemisphere of radius* a *with upward pointing normal vector* $\mathbf{n}$*.*

Solution. S isn't closed, but we would like to use the divergence theorem since

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x + yz^2) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(z) = 2$$

is so simple. Let S' be the unit disk in the x-y plane, with downward pointing normal $\mathbf{n} = -\mathbf{k}$. Then the total surface $S + S'$ *is* closed (and properly oriented with its normal vectors pointing outward), so we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dS = \oiint_{S+S'} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV = 2(\text{Vol } V) = \frac{4}{3}\pi a^3$$

[9] A surface is closed if there is no curve C serving as its boundary. To determine whether a surface is closed, use the "balloon test:" if the surface were a balloon filled with air, would it deflate? if so, the surface is not closed. The surfaces of the sphere and the torus are closed, for example, whereas the surface in figure 4 is not; the air would escape out the bottom through C.

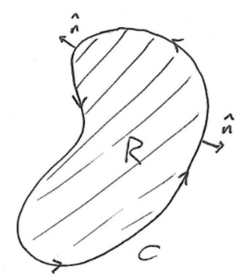

Figure 6: The relation between R and the normal vector $\mathbf{n}$ of its boundary. R is "to the left" of C, and $\mathbf{n}$ points *away* from R.

since V is the (solid) upper unit hemisphere. It remains to calculate $\iint_{S'} \mathbf{F}\cdot\mathbf{n}\, dS$, but

$$\mathbf{F}\cdot\mathbf{n} = \big((x+yz^2)\,\mathbf{i} + x^2z\,\mathbf{j} + z\,\mathbf{k}\big)\cdot(-\mathbf{k}) = z$$

which vanishes on the x-y plane where $z = 0$. Thus,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \frac{4}{3}\pi a^3$$

□

4 Extended Versions of the Theorems

The theorems above (with the exception of the Fundamental Theorem), have "extended versions," which allow us to consider regions *whose boundary consists of more than one closed curve/surface.*

In 2D, for example, if the region R is as in figure 7, Green's Theorem (in either form) still holds if we consider $C = \partial R$ to be the union of the three closed curves $C = C_1 + C_2 + C_3$. It is crucial to keep the orientation convention, however, so note that C_2 and C_3 are oriented *clockwise*, with normal vectors that point *away* from the region R.

An extended version of one of the line-to-area theorems will then look like:

$$\oint_{\partial R} (\ldots)\, ds = \sum_i \oint_{C_i} (\ldots)\, ds = \iint_R (\ldots)\, dA$$

The same is true in 3D; if D has boundary given by multiple closed surfaces, $\partial D = \sum_i S_i$, then the extended version of the divergence theorem will read

$$\oiint_{\partial D} \mathbf{F}\cdot\mathbf{n}\, dS = \sum_i \oiint_{S_i} \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_D \nabla\cdot\mathbf{F}\, dV$$

where the surfaces are oriented so that $\mathbf{n}$ points *away* from the interior R.

Remark. The proof of the extended version of any of the theorems can be deduced from the simply connected version by adding additional curves/surfaces to divide the region into multiple simply connected regions as in figure 7. The sum of the integrals over the simply connected regions will be equal to the integral over the original region, with the integrals over the added curves/surfaces canceling due to orientation.

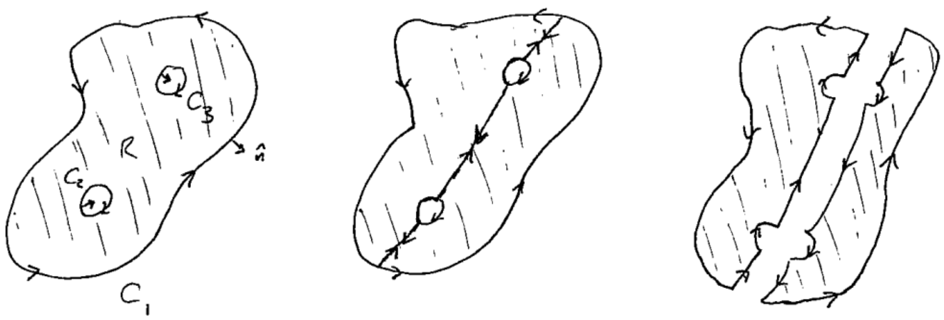

Figure 7: The total boundary of a non-simply connected region R. Note the orientation of C_1, C_2 and C_3. Splitting R into two regions indicates the proof of the extended version of the theorems.

Example. *Show that the flux of* $\mathbf{F} = 1/\rho^3\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = \hat{\mathbf{r}}/\rho^2$ *outward across any closed surface containing the origin is equal to* 4π*. (This is Gauss' Law for electromagnetism).*

Solution. Call our surface S. We would like to use the divergence theorem to calculate the flux. We get (using the formulas $\partial\rho/\partial x = x/\rho$, $\partial\rho/\partial y = y/\rho$, etc.)

$$\nabla \cdot \mathbf{F} = \frac{\rho^3 - 3x^2\rho}{\rho^6} + \frac{\rho^3 - 3y^2\rho}{\rho^6} + \frac{\rho^3 - 3z^2\rho}{\rho^6} = \frac{3}{\rho^3} - 3\frac{\rho^3}{\rho^6} = 0 \qquad \text{for } \rho \neq 0$$

but $\nabla \cdot \mathbf{F}$ is not defined at $\rho = 0$. Nevertheless, we can add a small sphere of radius ϵ around the origin and call this S_ϵ. Then $\nabla \cdot \mathbf{F} = 0$ *is* well-defined in the region D *between* S and S_ϵ. From the generalized version of the divergence theorem,

$$\oiint_{S+S_\epsilon} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV = 0 \Longrightarrow \oiint_S \mathbf{F}\cdot\mathbf{n}\,dS = -\oiint_{S_\epsilon} \mathbf{F}\cdot\mathbf{n}\,dS$$

where $\mathbf{n}$ points *inward*[10] towards the origin on S_ϵ.

So we can reduce the problem to the explicit calculation of the integral over S_ϵ. On this surface, $\mathbf{F}$ points in exactly the opposite direction to $\mathbf{n}$, so

$$\mathbf{F}\cdot\mathbf{n} = -\left|\mathbf{F}\right| = -\frac{1}{\rho^2} = -\frac{1}{\epsilon^2}$$

The surface element is given in spherical coordinates (see section 1.5.1) by

$$dS = \rho^2 \sin\phi\,d\phi\,d\theta = \epsilon^2 \sin\phi\,d\phi\,d\theta$$

so

$$\oiint_{S_\epsilon} \mathbf{F}\cdot\mathbf{n}\,dS = -\int_0^{2\pi}\int_0^{\pi} \frac{\epsilon^2}{\epsilon^2}\sin\phi\,d\phi\,d\theta = 2\pi\left(\cos\phi\right)\Big\|_{\phi=0}^{\pi} = -4\pi$$

Then by the above,

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,dS = -\oiint_{S_\epsilon} \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi$$

□

[10]In order to be consistent with the orientation convention, $\mathbf{n}$ must point *away* from D, which is inwards on S_ϵ.

5 Formulas and Theorems - Memorize!

Theorems for Work Integrals	Conditions
Stokes'/Green's Theorem (Flux form): $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$	∂S is one or more *closed* curves which bound the surface/region S. (In 2D, $S = R$, $\mathbf{n} = \mathbf{k}$ and $\nabla \times \mathbf{F} = (\nabla \times \mathbf{F})\ \mathbf{k}$)
FTCLI: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(p_1) - f(p_0)$	$\nabla \times \mathbf{F} = 0$ in D and D is *simply connected*, so $\mathbf{F} = \nabla f$. C must be entirely contatined in D with endpoints p_0 and p_1

Theorems for Flux Integrals	Conditions
Green's Theorem: $\oint_{\partial R} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$	∂R is a one or more *closed* curves which bound the region R.
Divergence Theorem: $\oiint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$	∂D is one or more *closed* surfaces which bound the domain D.

Forumlas for Parametrization & Differentiation	
$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$	For all your div ($\nabla \cdot \mathbf{F}$), curl ($\nabla \times \mathbf{F}$) and grad (∇f) needs.
$d\mathbf{r} = \mathbf{T}\, ds = dx\,\mathbf{i} + dy\,\mathbf{j} = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right) dt$	For work integrals in 2D.
$d\mathbf{r} = \mathbf{T}\, ds = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k} = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right) dt$	For work integrals in 3D.
$\mathbf{n}\, ds = dy\,\mathbf{i} - dx\,\mathbf{j} = \left(\frac{dy}{dt}\mathbf{i} - \frac{dx}{dt}\mathbf{j}\right) dt$	For flux integrals in 2D. (Note that $\mathbf{n}\, ds$ is just $d\mathbf{r}$, rotated 90° clockwise)
$\mathbf{n}\, dS = (-z_x\,\mathbf{i} - z_y\,\mathbf{j} + \mathbf{k})\ dx\, dy$	For flux integrals in 3D when S given by $z = z(x, y)$.
$\mathbf{n}\, dS = \frac{\nabla f}{f_z}\, dx\, dy$	For flux integrals in 3D when S given by $f(x, y, z) = 0$.
$dS = \sqrt{z_x^2 + z_y^2 + 1}\, dx\, dy$	For surface integrals when S given by $z = z(x, y)$. (Note this and the next follow from the above formulas for $\mathbf{n}\, dS$ by taking the magnitude: $dS = \lvert\mathbf{n}\, dS\rvert$)
$dS = \frac{1}{\lvert\mathbf{n} \cdot \mathbf{k}\rvert}\, dx\, dy = \left\lvert \frac{\nabla f}{f_z} \right\rvert\, dx\, dy$	For surface integrals when S given by $f(x, y, z) = 0$.

Cylindrical/Polar Coordinates	Spherical Coordinates
$r = \left(x^2 + y^2\right)^{1/2}$ $x = r\cos\theta$ $y = r\sin\theta$ $z = z$ $dA = r\, dr\, d\theta = (dr)\,(r\, d\theta)$ $dV = r\, dz\, dr\, d\theta = (dz)\,(dr)\,(r\, d\theta)$	$\rho = \left(x^2 + y^2 + z^2\right)^{1/2}$ $r = \rho\sin\phi$ $z = \rho\cos\phi$ $x = r\cos\theta = \rho\sin\phi\cos\theta$ $y = r\sin\theta = \rho\sin\phi\sin\theta$ $dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = (d\rho)\,(\rho\, d\phi)\,(\rho\sin\phi\, d\theta)$