EXTENSION OFF THE BOUNDARY IN A MANIFOLD WITH CORNERS

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Theorem 1. *Let M be a manifold with corners, and suppose $\{E_H : H \in \mathcal{M}_1(M)\}$ is a collection of vector bundles $E_H \longrightarrow H$ of fixed rank over the boundary hypersurfaces such that there are given isomorphisms*

$$E_H\big|_{H\cap H'} \cong E_{H'}\big|_{H\cap H'} \tag{1}$$

at the corners for each nonempty intersection $H \cap H'$. Then:

(a) *There exists an open neighborhood $U \supset \partial M$ of the total boundary of M and a vector bundle $E \longrightarrow U$ whose restriction to each $H \in \mathcal{M}_1(M)$ is isomorphic to E_H.*

(b) *If $\{\varphi_H \in C^\infty(H; E_H)\}$ is a collection of smooth sections of the E_H which are identified at corners with respect to* (1)*, i.e.*

$$\varphi_H\big|_{H\cap H'} \cong \varphi_{H'}\big|_{H\cap H'},$$

then there exists a section $\varphi \in C^\infty(U; E)$ whose restriction to each H agrees with φ_H.

(c) *If $\left\{\nabla^H\right\}$ is a collection of smooth connections on the E_H which are identified at corners with respect to* (1)*, then there exists a connection ∇ on $E \longrightarrow U$ whose restriction to each H agrees with ∇^H.*

Remark. In particular, if the E_H are trivial line bundles, part (b) gives an extension $f \in C^\infty(U)$ of a collection of smooth functions

$$\left\{f_H \in C^\infty(H) : H \in \mathcal{M}_1(M),\ f_H\big|_{H\cap H'} = f_{H'}\big|_{H\cap H'}\right\}.$$

The basic idea behind this result is simple. Near any boundary face, one takes f to be the sum of the pullbacks from the neighboring codimension 1 boundary faces, minus the sum of the pullbacks from the codimension 2 boundary faces, plus the sum of the pullbacks from the codimension 3 faces, and so on; see (4) below.

Proof of Theorem 1.(a). For each proper $G \in \mathcal{M}(M)$, let $E_G \longrightarrow G$ be given by the restriction of E_H for some $H \supset G$; this is well-defined up to isomorphism by the assumption on the E_H. On any product type neighborhood $V_{G,H} \cong G \times [0,1)^k$ of G in H (here k is the codimension of G in H) there is an isomorphism

$$(E_H)\big|_{V_{G,H}} \cong \pi_G^* E_G, \tag{2}$$

which follows from the smooth homotopy equivalence $V_{G,H} \sim G$.

For each $H \in \mathcal{M}_1(M)$, let $U_H \cong H \times [0,1)$ be a product type neighborhood of H in M, and for general $G \in \mathcal{M}(M)$, let

$$\widetilde{E}_G = \pi_G^* E_G \longrightarrow U_G, \text{ where}$$
$$U_G = \bigcap_{G\subset H\in\mathcal{M}_1(M)} U_H \cong G \times [0,1)^{\operatorname{codim}(G)}.$$

For $G \subset H$, it follows from (2) that there are isomorphims $\widetilde{E}_H \cong \widetilde{E}_G$ on $U_G \cap U_H$, and therefore $\widetilde{E}_H \cong \widetilde{E}_{H'}$ on $U_H \cap U_{H'} = U_{H\cap H'}$ for any $H, H' \in \mathcal{M}_1(M)$. It follows that the $\widetilde{E}_H \longrightarrow U_H$ patch together to form a smooth bundle

$$E \longrightarrow U := \bigcup_{H\in\mathcal{M}_1(M)} U_H. \qquad \square$$

For the extension of sections and connections, we first prove a local result. Let Y be a general manifold without boundary (possibly noncompact) and consider the product

$$X = Y \times [0,1)^n.$$

For each $I \subset \{1,\ldots,n\}$, there is an associated boundary face $B_I \in \mathcal{M}_{|I|}(X)$ along with an inclusion and a projection:

$$\begin{gathered} B_I = \{(y,x_1,\ldots,x_n) \in X : x_i = 0,\ \forall\, i \in I\}, \\ \iota_I : B_I \hookrightarrow X, \quad \pi_I : X \longrightarrow B_I, \\ \pi_I \circ \iota_I = \mathrm{Id} : B_I \longrightarrow B_I. \end{gathered}$$

We use the notation B_i instead of $B_{\{i\}}$ for boundary hypersurfaces. Suppose $E \longrightarrow X$ is a vector bundle, which without loss of generality (composing with an isomorphism if necessary) we may assume is of the form $\pi_Y^* E_Y = \pi^*_{\{1,\ldots,n\}} E_Y$ for a bundle $E_Y \longrightarrow Y$.

Lemma 2. *If $\{\varphi_i \in C^\infty(B_i;E) : 1 \le i \le n\}$ is a collection of smooth sections of E on the boundary hypersurfaces which agree at all corners, i.e.*

$$(\varphi_i)\big|_{B_I} = (\varphi_j)\big|_{B_I},\ \forall\, I \supset \{i,j\}, \tag{3}$$

then there is a section $\varphi \in C^\infty(X;E)$ whose restriction to each of the B_i agrees with φ_i.

Proof. For each $I \subset \{1,\ldots,n\}$, define $\varphi_I \in C^\infty(B_I)$ by restriction of some φ_i where $i \in I$. By (3), this does not depend on the choice of $i \in I$. Then define f by

$$\varphi = \sum_{1\le|I|} (-1)^{|I|+1} \pi_I^* \varphi_I. \tag{4}$$

Since $E = \pi_I^*(\pi_Y^* E_Y)$ for each I, such an expression is well-defined. To see that $\varphi\big|_{B_i} = \varphi_i$, we consider the pullback of (4) by ι_i, and note that

$$\pi_I \circ \iota_i = \pi_{I\cup\{i\}} : B_i \longrightarrow B_{I\cup\{i\}}.$$

Thus,

$$\begin{aligned} \iota_i^* \varphi &= \sum_{1\le|I|} (-1)^{|I|+1} \pi^*_{I\cup\{i\}} \varphi_{I\cup\{i\}} \\ &= \varphi_i + \sum_{\substack{1\le|I| \\ I\neq\{i\}}} (-1)^{|I|+1} \pi^*_{I\cup\{i\}} \varphi_{I\cup\{i\}} \\ &= \varphi_i + \sum_{\substack{1\le|I| \\ i\notin I}} \big((-1)^{|I|+1} \pi^*_{I\cup\{i\}} \varphi_{I\cup\{i\}} + (-1)^{|I\cup\{i\}|+1} \pi^*_{I\cup\{i\}} \varphi_{I\cup\{i\}} \big) \\ &= \varphi_i. \quad \square \end{aligned}$$

Proof of Theorem 1, (b) *and* (c). For each proper boundary face $G \in \mathcal{M}(M)$, denote by $\varphi_G \in C^\infty(M; E_G)$ the restriction to G of φ_H for some $H \in \mathcal{M}_1(M)$ such that $G \subset H$, which is independent of the choice of H by the compatibility of the φ_H at corners. Let

$$U'_G \cong \mathring{G} \times [0,1)^{\operatorname{codim}(G)}$$

be an open product-type neighborhood of the *interior* of G (in contrast to U_G defined above), set

$$U' = \bigcup_{G \in \mathcal{M}(M)} U'_G,$$

and let $\{\chi_G\}$ be a partition of unity on U' subordinate to the cover $\{U'_G\}$. Since $\mathring{G} = G$ for corners of maximal codimension, it can always be arranged that $U' = U$, where U is the neighborhood from part (a). Let $E \longrightarrow U'$ be the bundle from (a), and regard each φ_G is a section of E over G.

On each U'_G let $\widetilde{\varphi}_G \in C^\infty(U'_G; E)$ be a section restricting to φ_H on $U'_G \cap H$ for each hypersurface H such that $G \subset H$, as in Lemma 2. Then

$$\varphi = \sum_{G \in \mathcal{M}(M)} \chi_G \widetilde{\varphi}_G \in C^\infty(U; E) \tag{5}$$

has the desired properties.

We claim that the same procedure allows the connections to be extended. Of course it does not make sense to take a general linear combination of connections since they do not form a vector space but rather an affine space modelled on (endomorphism-valued) one-forms. However, provided the coefficients in the linear combination sum identically to 1, such a linear combination in an affine space makes sense. Indeed, if $\{\nabla_1, \ldots, \nabla_n\}$ are connections on a given bundle $E \longrightarrow X$, and $\{a_1, \ldots, a_n : a_i \in C^\infty(X)\}$ are functions such that $\sum_i a_i \equiv 1$, then we may take as a definition

$$a_1 \nabla_1 + \cdots + a_n \nabla_n := \nabla_1 + a_2(\nabla_2 - \nabla_1) + \cdots + a_n(\nabla_n - \nabla_1), \tag{6}$$

where $(\nabla_i - \nabla_1) \in \Omega^1(X; \operatorname{End}(E))$ denotes the one-form α_i such that $\nabla_1 + \alpha_i = \nabla_i$. Since $a_1 = 1 - \sum_{i=2}^n a_i$, (6) is well-defined independent of the ordering of the ∇_i.

It then suffices to note that the coefficients in (4) sum to the identity, since

$$\sum_{1 \le |I|} (-1)^{|I|+1} = - \sum_{1 \le k \le n} \binom{n}{k} (-1)^k = 1 - (1-1)^n = 1,$$

and of course the coefficients in (5) sum to the identity by construction. □

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