EXT, TOR AND THE UCT

CHRIS KOTTKE

Contents

1. Left/right exact functors

We start with the following observation.

Proposition 1.1. *For fixed $G \in \mathsf{AbGp}$, if $0 \to A \to A' \to A'' \to 0$ is a short exact sequence of abelian groups, then*

$$0 \longrightarrow \mathrm{Hom}(A'', G) \longrightarrow \mathrm{Hom}(A', G) \longrightarrow \mathrm{Hom}(A, G)$$

is exact.

We say the functor $\mathrm{Hom}(-, G)$ is only *left exact.* In general we have the following definition.

Definition 1.2. Let $F : \mathsf{AbGp} \longrightarrow \mathsf{AbGp}$ be a contravariant functor, and let

$$0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0 \tag{1}$$

be a short exact sequence. If the corresponding sequence

$$0 \longrightarrow F(A'') \longrightarrow F(A') \longrightarrow F(A) \tag{2}$$

is exact, we say F is *left exact.* If instead

$$F(A'') \longrightarrow F(A') \longrightarrow F(A) \longrightarrow 0$$

we say F is *right exact.* If F is both right and left exact, we say it is *exact*; equivalently F is exact if

$$0 \longrightarrow F(A'') \longrightarrow F(A') \longrightarrow F(A) \longrightarrow 0$$

is exact.

Similarly, if F is a covariant functor, we say it is left exact if

$$0 \longrightarrow F(A) \longrightarrow F(A') \longrightarrow F(A'')$$

is exact, and so on.

The theory of *derived functors* (see [Wei95]) gives a mechanism for computing (in principle) a continuation of (2) to an exact sequence

$$\begin{aligned} 0 \longrightarrow F(A) \longrightarrow F(A') \longrightarrow F(A'') \longrightarrow R^1F(A) \\ \longrightarrow R^1F(A') \longrightarrow R^1F(A'') \longrightarrow R^2F(A) \longrightarrow \cdots \end{aligned}$$

where the objects $R^iF(A)$ are known as the ith *right derived functors* of F.

We will not pursue the completely general theory[1] but rather stick to the case of $\mathrm{Hom}(-, G)$ and $- \otimes G$ in the category of abelian groups, whose derived functors go by the names of $\mathrm{Ext}^*(-, G)$ and $\mathrm{Tor}^*(-, G)$, respectively. However, we will nevertheless go through the "proper" construction of these objects, to get the essence of the theory.

2. Projective resolutions

The main tool in this theory is the notion of a "projective" object, which in essence allows us to infer the existence of lifting maps in particular situations.

Definition 2.1. We say an abelian group P is *projective* if given a homomorphism $f : P \longrightarrow G$ and a surjective homomorphism $h : G' \longrightarrow G$, there exists a lift $f' : P \longrightarrow G'$ such that

$$\begin{array}{ccccc} & & P & & \\ & \overset{\exists f'}{\swarrow} & \downarrow f & & \\ G' & \xrightarrow{h} & G & \longrightarrow & 0 \end{array} \tag{3}$$

commutes.

Remark. The defining property of a projective object is a categorical one, and makes sense in any so-called "abelian category" where things like kernels, cokernels and exactness make sense. For instance, we can define projective objects in categories of chain complexes, modules over a ring R, sheaves of such objects and so on.

Definition 2.2. Fix $A \in \mathsf{AbGp}$. A *projective resolution* of A is an exact sequence

$$\cdots \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{d} A \longrightarrow 0$$

where the objects P_i are projective.

It turns out that there are particularly simple resolutions for any abelian group using free groups.

Lemma 2.3. *Any free abelian group is projective. Furthermore, for any $A \in \mathsf{AbGp}$, there is a two step projective resolution*

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

where R and F are free groups.

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[1] which can be very general indeed, leading eventually to the theory of so-called "derived categories" which are of importance in algebraic geometry.

Proof. For the first claim, suppose P is free. It suffices to define f' on generators $\{p_i\}_{i\in I}$ of P, and we can set

$$f(p_i) = g_i', \quad \text{for any } g_i' \in h^{-1}\big(f(p_i)\big).$$

For the second claim, let F be the free abelian group on any set of generators for A. Then F clearly surjects onto A and we let R be its kernel. It is then a standard fact that any subgroup of a free group is free, so R is also free. □

Because of this result, we could restrict ourselves to free resolutions, and two step ones at that. Nevertheless, we will continue to use projective resolutions to illustrate general ideas.

3. Two useful lemmas

The general yoga of derived functors is to execute the following steps

(1) Take a projective resolution

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

of our given object.

(2) Chop off the A from the end to obtain a (no longer exact) sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

This will serve in a certain sense as a substitute for A. Observe that it is still a complex since the composition of two maps is 0.

(3) Apply the left exact functor $F(-) = \mathrm{Hom}(-, G)$ (say, for definiteness) to get

$$0 \longrightarrow F(P_0) \longrightarrow F(P_1) \longrightarrow F(P_2) \longrightarrow \cdots$$

which is a complex since F takes the 0 morphism to the 0 morphism.

(4) Compute the (co)homology groups of this complex to get

$$R^i F(A) := \frac{\mathrm{Ker}\,\{F(P_i) \longrightarrow F(P_{i+1})\}}{\mathrm{Im}\,\{F(P_{i-1}) \longrightarrow F(P_i)\}}$$

these are the *right derived functors* of F applied to A. We will observe that $R^0F(A) \equiv F(A)$, i.e. the 0th derived functor of F just gives us F back.

(5) Observe that for any exact sequence

$$0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0 \tag{4}$$

there is a long exact sequence

$$\begin{aligned} 0 \longrightarrow F(A'') \longrightarrow F(A') \longrightarrow F(A) \longrightarrow R^1F(A'') \\ \longrightarrow R^1F(A') \longrightarrow R^1F(A) \longrightarrow R^2F(A'') \longrightarrow \cdots \end{aligned} \tag{5}$$

In order to make this go through, we need to verify that the result is independent of the projective resolution that we choose, and that we can make the projective resolutions for (4) fit together into a (split) short exact sequence of complexes, so that we get the long exact sequence (5) when we take homology. We'll prove these things in the following two lemmas. Observe that the proofs are mostly applications of the projectivity property (3), and as such are valid in a much more general context.

Lemma 3.1. *Let* $f : A \longrightarrow B$ *be a homomorphism of abelian groups and let*

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

and

$$\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow B \longrightarrow 0$$

be projective resolutions. Then there is a chain map $f_* : (P_* \longrightarrow A) \longrightarrow (Q_* \longrightarrow B)$ *extending* f*, in other words a sequence of homomorphisms* $f_i : P_i \longrightarrow Q_i$ *such that*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

commutes. Furthermore, any two such extensions of f *are chain homotopic.*

Proof. To prove existence of the f_i, we proceed by induction on i, considering the given map $f_{-1} := f : A \longrightarrow B$ as the base case. Thus assume that $f_i : P_i \longrightarrow Q_i$ has been constructed. Denoting by $Z_i(P_i)$ and $Z_i(Q_i)$ the cycles in P_i and Q_i respectively, observe that $f_i : Z_i(P_i) \longrightarrow Z_i(Q_i)$ since $f_{i-1}\, d = d\, f_i$. Thus consider the diagram

$$\begin{array}{ccccc} P_{i+1} & \xrightarrow{\;d\;} & Z_i(P_i) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \\ Q_{i+1} & \xrightarrow{\;d\;} & Z_i(Q_i) & \longrightarrow & 0 \end{array}$$

The composition $f_i\, d$ gives a map from P_{i+1} to $Z_i(Q_i)$, onto which Q_{i+1} surjects. Thus the requisite map f_{i+1} is furnished by the defining property (3) of the projective group P_{i+1}, completing the induction.

To show that two extensions $\{f_i\}$ and $\{f_i'\}$ are chain homotopic, we consider the difference $g_i := f_i - f_i'$ and construct a chain homotopy $\{s_i : P_i \longrightarrow Q_{i+1}\}$ such that $g = ds + sd$. Again we proceed by induction. Observe that $g_{-1} = f - f \equiv 0$, so that g_0 maps P_0 into cycles $Z_0(Q_0)$, and therefore lifts to a map $s_0 : P_0 \longrightarrow Q_1$ as in the following diagram:

$$\begin{array}{ccccc} & & P_0 & \longrightarrow & A \\ & \swarrow{\scriptstyle s_0} & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_{-1} \equiv 0} \\ Q_1 & \longrightarrow & Z_0(Q_0) & \longrightarrow & 0 \end{array}$$

This gives the base case for the induction.

Suppose then that $s_i : P_i \longrightarrow Q_{i+1}$ has been constructed such that $g_i = s_{i-1}\, d + d\, s_i$, or equivalently

$$d\, s_i = g_i - s_{i-1}\, d.$$

It follows that the map $g_{i+1} - s_i\, d$ maps P_{i+1} into cycles $Z_{i+1}(Q_{i+1})$ since

$$d\big(g_{i+1} - s_i\, d\big) = g_i\, d - \big(g_i - s_{i-1}\, d\big)d = g_i\, d - g_i\, d = 0.$$

Thus we have the diagram

$$\begin{array}{ccccc} & & P_{i+1} & & \\ & \swarrow^{s_{i+1}} & \Big\downarrow g_{i+1} - s_i\, d & & \\ Q_{i+2} & \longrightarrow & Z_{i+1}(Q_{i+1}) & \longrightarrow & 0 \end{array}$$

and projectivity furnishes the map s_{i+1} such that $d\, s_{i+1} = g_{i+1} - s_i\, d$. ☐

Lemma 3.2. *Let $0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ be a short exact sequence. Then there is a short exact sequence of projective resolutions $P_* \longrightarrow A$, $P'_* \longrightarrow A'$ and $P''_* \longrightarrow A''$ such that*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

commutes and the exact sequences $0 \longrightarrow P_i \longrightarrow P'_i \longrightarrow P''_i \longrightarrow 0$ are split.

Proof. Choose any projective resolutions $P_* \longrightarrow A$ and $P''_* \longrightarrow A''$, giving a diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & A' & & \\ & & & & & & \downarrow & & \\ \cdots & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \longrightarrow & A'' & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

We will fill in the middle with a projective resolution satisfying the claimed properties. (For this reason, this lemma is sometimes referred to as the "horseshoe lemma".)

Composition gives a map $P_0 \longrightarrow A'$, and a map $P''_0 \longrightarrow A'$ is furnished by projectivity. These combine to give a map $P_0 \oplus P''_0 \longrightarrow A'$, and we set $P'_0 := P_0 \oplus P''_0$, obtaining the diagram

$$\begin{array}{ccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
& & P_0 \oplus P_0'' & \longrightarrow & A' & & \\
& & \downarrow & & \downarrow & & \\
& & P_0'' & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

The sequence $P_0 \longrightarrow P_0 \oplus P_0'' \longrightarrow P_0''$ is obviously split exact; we will show that the map $P_0 \oplus P_0'' \longrightarrow A'$ is surjective.

Fix $0 \neq \alpha' \in A'$. There are two cases to consider. If $\alpha' \in \operatorname{Ker}\{A' \longrightarrow A''\}$, then it is in the image of the map from A, and therefore also in the image of the map from P_0. If $\alpha' \notin \operatorname{Ker}\{A' \longrightarrow A''\}$ then it has nonzero image $\alpha'' \in A''$, and there is some $p'' \in P_0''$ mapping onto α''. By commutativity of (3), this p'' maps onto α', so every element in A' is either in the image of P_0 or P_0''.

The proof continues by induction, replacing $0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ by the short exact sequence of cycles $0 \longrightarrow Z_i(P_i) \longrightarrow Z_i(P_i') \longrightarrow Z_i(P_i'') \longrightarrow 0$. □

4. Ext

We are now ready to construct the groups $\operatorname{Ext}^i(A, G)$ as derived functors of the functor $A \longmapsto \operatorname{Hom}(A, G)$.

Choose a projective resolution

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

and apply $\operatorname{Hom}(-, G)$ to the truncated sequence $P_* \longrightarrow 0$, obtaining

$$0 \longrightarrow \operatorname{Hom}(P_0, G) \longrightarrow \operatorname{Hom}(P_1, G) \longrightarrow \cdots \tag{6}$$

which is a chain complex.

Definition 4.1. The group $\operatorname{Ext}^i(A, G)$ is the ith (co)homology group of (6):

$$\operatorname{Ext}^i(A, G) := \frac{\operatorname{Ker}\{\operatorname{Hom}(P_i, G) \longrightarrow \operatorname{Hom}(P_{i+1}, G)\}}{\operatorname{Im}\{\operatorname{Hom}(P_{i-1}, G) \longrightarrow \operatorname{Hom}(P_i, G)\}} \tag{7}$$

First we make some observations that are valid for right derived functors in general.

Proposition 4.2. *The groups* $\operatorname{Ext}^i(A, G)$ *are independent of the projective resolution of* A.

Proof. Let $P_* \longrightarrow A$ and $Q_* \longrightarrow A$ be two projective resolutions. Lemma 3.1 gives extensions of the identity map $\operatorname{Id} : A \longrightarrow A$ to chain maps $f : P_* \longrightarrow Q_*$ and $g : Q_* \longrightarrow P_*$. Observe that both $g\,f : P_* \longrightarrow P_*$ and $\operatorname{Id} : P_* \longrightarrow P_*$ extend the identity on A, and are therefore chain homotopic, via some $s : P_* \longrightarrow P_{*+1}$.

Applying the functor $\mathrm{Hom}(-,G)$, we obtain maps $f^* : \mathrm{Hom}(Q_*,G) \longrightarrow \mathrm{Hom}(P_*,G)$, $g^* : \mathrm{Hom}(P_*,G) \longrightarrow \mathrm{Hom}(Q_*,G)$ and $s^* : \mathrm{Hom}(P_*,G) \longrightarrow \mathrm{Hom}(P_{*-1},G)$, and the identity

$$f^*\, g^* - \mathrm{Id} = s^*\, \delta + \delta s^*$$

holds where $\delta := d^*$, since $\mathrm{Id}^* = \mathrm{Id}$. Thus the maps f^* and g^* induce isomorphisms on homology groups. □

Proposition 4.3. *The group* $\mathrm{Ext}^0(A,G)$ *is canonically isomorphic to* $\mathrm{Hom}(A,G)$:

$$\mathrm{Ext}^0(A,G) \cong \mathrm{Hom}(A,G), \quad \forall\, A, G$$

Proof. By left exactness, the sequence

$$0 \longrightarrow \mathrm{Hom}(A,G) \longrightarrow \mathrm{Hom}(P_0,G) \longrightarrow \mathrm{Hom}(P_1,G)$$

is exact. Thus the zeroth homology group $\mathrm{Ext}^0(A,G)$, which is just the kernel of the first map $\mathrm{Hom}(P_0,G) \longrightarrow \mathrm{Hom}(P_1,G)$ in (6), is identified with $\mathrm{Hom}(A,G)$. □

Proposition 4.4. *For any projective group P, all the higher Ext groups vanish:*

$$\mathrm{Ext}^i(P,G) \equiv 0, \quad i \geq 1$$

Proof. The sequence

$$0 \longrightarrow P \xrightarrow{\mathrm{Id}} P \longrightarrow 0$$

is a projective resolution of P! □

Proposition 4.5. *If* $0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ *is an exact sequence, there is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(A'',G) \longrightarrow \mathrm{Hom}(A',G) \longrightarrow \mathrm{Hom}(A,G) \\ \longrightarrow \mathrm{Ext}^1(A'',G) \longrightarrow \mathrm{Ext}^1(A',G) \longrightarrow \mathrm{Ext}^1(A,G) \longrightarrow \cdots \end{aligned} \tag{8}$$

Proof. Choose projective resolutions according to Lemma 3.2. Since the sequences $0 \longrightarrow P_i \longrightarrow P_i' \longrightarrow P_i'' \longrightarrow 0$ are split exact, it follows that

$$0 \longrightarrow \mathrm{Hom}(P_*'',G) \longrightarrow \mathrm{Hom}(P_*',G) \longrightarrow \mathrm{Hom}(P_*,G) \longrightarrow 0$$

is a short exact sequence of complexes. (*A priori* it is only left exact, but then it follows from the splitting that it is also exact on the right.) The sequence (8) then follows from the usual long exact sequence in homology and Proposition 4.3. □

Finally, we have the following results, which are particular to the $\mathrm{Hom}(-,G)$ functor and/or the category of abelian groups. The proof of the next Proposition follows directly from Lemma 2.3.

Proposition 4.6. *For any abelian groups A and G, the only nonzero Ext groups are in degrees 0 and 1:*

$$\mathrm{Ext}^i(A,G) = 0, \quad i \geq 2.$$

For this reason, the groups $\mathrm{Ext}^1(A, G)$ are often just denoted $\mathrm{Ext}(A, G)$. Note however that this construction can be applied to the setting of modules over a ring R (generalizing abelian groups which are equivalent to modules over $\mathbb{Z}$), and then the groups $\mathrm{Ext}^i_R(M, N)$, $i \geq 2$ are nonzero in general.

Proposition 4.7. $\mathrm{Ext}^i(\bigoplus_\alpha A_\alpha, G) = \bigoplus_\alpha \mathrm{Ext}^i(A_\alpha, G)$ *and* $\mathrm{Ext}^i(A, \prod_\alpha G_\alpha) = \prod_\alpha \mathrm{Ext}^i(A, G_\alpha)$

Proof. These follow directly from the corresponding identities

$$\mathrm{Hom}(\bigoplus_\alpha A_\alpha, G) = \bigoplus_\alpha \mathrm{Hom}(A_\alpha, G), \text{ and}$$
$$\mathrm{Hom}(A, \prod_\alpha G_\alpha) = \prod_\alpha \mathrm{Hom}(A, G_\alpha).$$

by and by taking direct sums of projectives to form a resolution for $\bigoplus_\alpha A_\alpha$. □

A particularly important computation is the following.

Proposition 4.8. $\mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$. *More generally, for any group G,* $\mathrm{Ext}^1(\mathbb{Z}_m, G) = G/mG$.

Proof. Use the resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow 0.$$

Then $\mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z})$ is the second cohomology group of

$$0 \longrightarrow \mathbb{Z} \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{m} \mathbb{Z} \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$$

which is $\mathbb{Z}_m$. Similarly, since $\mathrm{Hom}(\mathbb{Z}, G) = G$, it follows that $\mathrm{Ext}^1(\mathbb{Z}_m, G) = G/mG$. □

Remark. The name Ext comes from "Extension." We say X is an *extension* of A by B if

$$0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$$

is exact. Given A and B there is always the trivial extension $X = A \oplus B$, corresponding to the isomorphism class of the split exact sequence. It can be shown (see [Wei95]) that isomorphism classes of extensions of A by B are in 1-1 correspondence with elements of $\mathrm{Ext}^1(A, B)$, with the trivial extension corresponding to 0.

5. Ext as a covariant derived functor

There is another way to define the groups $\mathrm{Ext}^1(A, G)$, namely, as the derived functors of the *covariant* functor

$$\mathrm{Hom}(A, -) : G \longmapsto \mathrm{Hom}(A, G).$$

Proposition 5.1. $\mathrm{Hom}(A, -)$ *is left exact. Thus if*

$$0 \longrightarrow G \longrightarrow G' \longrightarrow G'' \longrightarrow 0$$

is exact, then

$$0 \longrightarrow \mathrm{Hom}(A, G) \longrightarrow \mathrm{Hom}(A, G') \longrightarrow \mathrm{Hom}(A, G'')$$

is exact.

We can perform almost the same procedure as before, though because of the combination of covariance and left exactness, the procedure gets dualized. In particular, we must consider *injective* objects instead of projective ones.

Definition 5.2. An abelian group I is *injective* if for any homomorphism $f : G \longrightarrow I$ and injective homomorphism $h : G \longrightarrow G'$, there exists a homomorphism $f' : G' \longrightarrow I$ such that

$$\begin{array}{ccccc} & & I & & \\ & \overset{\exists f'}{\nearrow} & \uparrow f & & \\ G' & \overset{h}{\longleftarrow} & G & \longleftarrow & 0 \end{array} \tag{9}$$

commutes.

Given $G \in \mathsf{AbGp}$, an *injective resolution* is an exact sequence

$$0 \longrightarrow G \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

where the groups I_j are injective.

Taking an injective resolution $G \longrightarrow I_*$ for G and applying the functor $\mathrm{Hom}(A, -)$ to the truncated complex $0 \longrightarrow I_*$ results in the complex

$$0 \longrightarrow \mathrm{Hom}(A, I_0) \longrightarrow \mathrm{Hom}(A, I_1) \longrightarrow \cdots .$$

Taking the cohomology of this sequence results in groups

$$\mathrm{Ext}^i(A, G) := \frac{\mathrm{Ker}\,\{\mathrm{Hom}(A, I_i) \longrightarrow \mathrm{Hom}(A, I_{i+1})\}}{\mathrm{Im}\,\{\mathrm{Hom}(A, I_{i-1}) \longrightarrow \mathrm{Hom}(A, I_i)\}} \tag{10}$$

which turn out to be the same as those defined by deriving the functor $\mathrm{Hom}(-, G)$.

Proposition 5.3. *The groups defined by* (10) *coincide with those defined by* (7)*.*

Proof. See [Wei95], Theorem 2.7.6. □

Again, in the category of abelian groups it turns out that there is always a two step injective resolution which gives another way to show that $\mathrm{Ext}^i(A, G) = 0$ for $i \geq 2$.

Furthermore, dualizing the lemmas in Section 3 (whose proofs are the same up to replacing "projective" by "injective" and reversing all arrows) shows that $\mathrm{Ext}^i(A, G)$ is independent of the injective resolution of G, and we have the following analogue of Proposition 4.5.

Proposition 5.4. *From a short exact sequence*

$$0 \longrightarrow G \longrightarrow G' \longrightarrow G'' \longrightarrow 0$$

we obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(A, G) \longrightarrow \mathrm{Hom}(A, G') \longrightarrow \mathrm{Hom}(A, G'') \\ \longrightarrow \mathrm{Ext}^1(A, G) \longrightarrow \mathrm{Ext}^1(A, G') \longrightarrow \mathrm{Ext}^1(A, G'') \longrightarrow 0 \end{aligned}$$

There is also the obvious analogue of Proposition 4.4.

Proposition 5.5. *If $I \in \mathsf{AbGp}$ is injective, then*

$$\mathrm{Ext}^i(A, I) = 0, \quad \forall A$$

6. Universal Coefficient Theorem for Cohomology

The universal coefficient theorem for cohomology quantifies the difference between the groups $H^n(X,G) = H^n\big(\mathrm{Hom}(C_*(X),G),\delta\big)$ and $\mathrm{Hom}(H_n(X),G)$ in terms of the groups $\mathrm{Ext}\big(H_{n-1}(X),G\big)$. Observe that there is a natural map

$$h : H^n(X,G) \longrightarrow \mathrm{Hom}\big(H_n(X),G\big), \quad \text{where} \quad h([f])([c]) = f(c).$$

Here $[f]$ denotes the cohomology class of a cocycle $f \in C^*(X,G) = \mathrm{Hom}(C_*(X),G)$, and $[c]$ denotes the homology class of a cycle $c \in C_*(X)$. The map does not depend on the choices of representative: since $\delta f = 0$ and $\partial c = 0$ it follows that

$$h([\delta g])([c]) = \delta g(c) = g(\partial c) = 0, \quad \text{and} \quad h([f])([\partial d]) = f(\partial d) = \delta f(d) = 0.$$

We will prove a completely algebraic version of the universal coefficient theorem first. Let (C_*,∂) be a chain complex of free groups, and denote by

$$Z_n = \{\alpha \in C_n : \partial\alpha = 0\}\,, \quad B_n = \{\partial\beta : \beta \in C_{n+1}\}$$

the cycles and boundaries of C_n, respectively. Additionally, we will use the shorthand H_n to denote the homology group $H_n(C_*),\partial)$.

We have short exact sequences

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0 \tag{11}$$

and

$$0 \longrightarrow B_n \xrightarrow{i} Z_n \xrightarrow{q} H_n \longrightarrow 0, \tag{12}$$

and we note that the groups B_n and Z_n are free, being subgroups of the free groups C_n. From the theory of Ext groups, these lead to exact sequences

$$0 \longrightarrow \mathrm{Hom}(B_{n-1},G) \xrightarrow{\delta} \mathrm{Hom}(C_n,G) \xrightarrow{i^*} \mathrm{Hom}(Z_n,G) \longrightarrow 0 \tag{13}$$

and

$$0 \longrightarrow \mathrm{Hom}(H_n,G) \xrightarrow{q^*} \mathrm{Hom}(Z_n,G) \xrightarrow{i^*} \mathrm{Hom}(B_n,G) \longrightarrow \mathrm{Ext}(H_n,G) \longrightarrow 0 \tag{14}$$

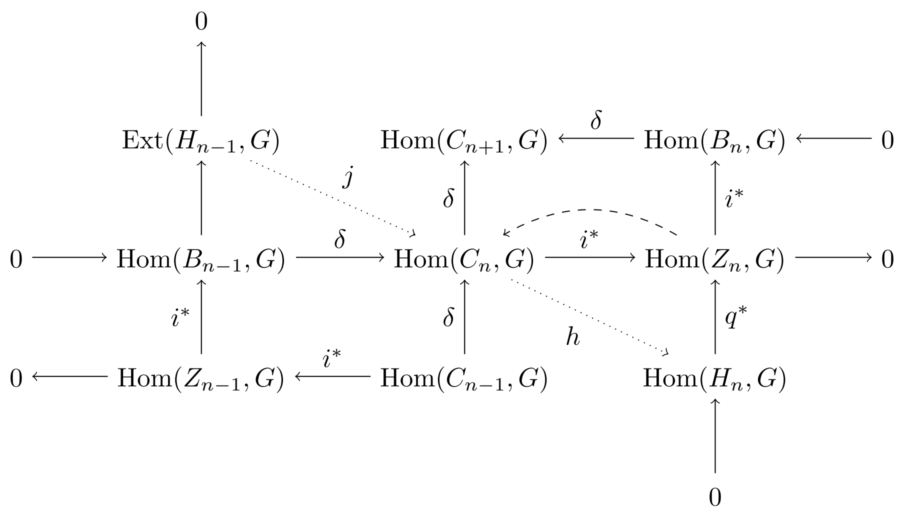
Note that since (11) is a sequence of free groups it splits, though not naturally with respect to the boundary maps $\partial : C_n \longrightarrow C_{n-1}$.

Theorem 6.1. *There are short exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1},G) \xrightarrow{j} H^n\big(\mathrm{Hom}(C_*,G)\big) \xrightarrow{h} \mathrm{Hom}(H_n,G) \longrightarrow 0 \tag{15}$$

for each n, which are natural in G and C_*, and which split (though not naturally with respect to C_*).

Proof. The proof (taken from [Bre97]) involves a bunch of chasing around the somewhat complicated diagram



A few notes about the diagram: The columns on the right and left are pieces of the sequences (14) in degrees $n-1$ and n, respectively. The middle column is just the chain complex for cohomology. Finally, the horizontal maps are pieces of the sequences (13) in degrees $n+1$, n and $n-1$, respectively, from top to bottom. The dashed map is induced by a choice of splitting $C_n \longrightarrow Z_n$ of the sequence (11). The dotted maps are the ones we are defining; they are not well-defined at the level of cochains, but we'll show they are well-defined on cohomology classes.

The map h which we defined earlier is equivalent to the following. It takes a cocycle $f \in \mathrm{Hom}(C_n, G)$ in the middle of the diagram and goes [right, down] (since $\delta f = 0$, [right, up] results in 0 so the resulting element of $\mathrm{Hom}(Z_n, G)$ is in the image of $\mathrm{Hom}(H_n, G)$). If $f = \delta g$ to begin with, then it comes from $\mathrm{Hom}(B_{n-1}, G)$ by commutativity (from g, going [up] is equivalent to [left, up, right]), and then going [right] is equivalent to a path involving two consecutive steps in the middle row, resulting in 0.

The map j taking $\mathrm{Ext}(H_{n-1}, G)$ to a cohomology class in $\mathrm{Hom}(C_n, G)$ is defined similarly, by lifting [down] and going [right]. The resulting element $f \in \mathrm{Hom}(C_n, G)$ satisfies $\delta f = 0$ since δ [up] is equivalent to [right, up, left] and that involves two steps in the middle row, giving 0. Similarly, the ambiguity in the initial lift [down] comes from $\mathrm{Hom}(Z_{n-1}, G)$, which maps into the image of δ in $\mathrm{Hom}(C_n, G)$ and so the image of j is well-defined in cohomology.

It is straightforward to see that composing j and h results in 0, as is injectivity of j and surjectivity of h.

To see exactness in the middle of (15), note that any $f \in \mathrm{Hom}(C_n, G)$ which goes to 0 by h must already vanish at $\mathrm{Hom}(Z_n, G)$, hence it lies in the image of $\mathrm{Hom}(B_{n-1}, G)$ and comes from some element in $\mathrm{Ext}(H_n, G)$ by j.

Naturality with respect to G and C_* will be left to the contemplation of the reader.

Finally, the splitting of (15) is induced by the dashed map, which is natural with respect to G but not with respect to the indexing by n (i.e. it is not natural with respect to chain maps such as $\partial : C_* \longrightarrow C_{*-1}$). □

Applying Theorem 6.1 to the chain complex $C_*(X)$, we obtain what is usually called the Universal Coefficient Theorem for Cohomology.

Corollary 6.2 (Universal Coefficient Theorem)**.** *There are short exact sequences*

$$0 \longrightarrow \mathrm{Ext}\big(H_{n-1}(X), G\big) \longrightarrow H^n(X; G) \longrightarrow \mathrm{Hom}\big(H_n(X), G\big) \longrightarrow 0 \tag{16}$$

for each n, which are natural in G and X, and which split (though not naturally with respect to X). There are similar short exact sequences

$$0 \longrightarrow \mathrm{Ext}\big(H_{n-1}(X, A), G\big) \longrightarrow H^n(X, A; G) \longrightarrow \mathrm{Hom}\big(H_n(X, A), G\big) \longrightarrow 0$$

for a pair (X, A).

If a space X has finitely generated homology in all degrees, then we can decompose its homology groups as

$$H_n(X) \cong F_n \oplus T_n$$

where $F_n = \bigoplus \mathbb{Z}$ is a maximal finitely generated free abelian subgroup, and the *torsion* T_n is a finite direct sum of finite cyclic groups. Combining this with Proposition 4.8, we obtain the following

Proposition 6.3. *If X has finitely generated homology, then*

$$H^n(X) \cong F_n \oplus T_{n-1} \tag{17}$$

where $H^n(X) = H^n(X, \mathbb{Z})$ denotes cohomology with $\mathbb{Z}$ coefficients.

Proof. Since $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0$, $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$ and $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$ it follows that

$$\mathrm{Hom}\big(H_n(X), \mathbb{Z}\big) \cong F_n, \quad \mathrm{Ext}\big(H_{n-1}(X), \mathbb{Z}\big) \cong T_{n-1}.$$

The isomorphism (17) then follows from the splitting of (16). □

7. Tensor product

We recall without proof some results about the tensor product.

Definition 7.1. If $A, B \in \mathsf{AbGp}$, the group $A \otimes B$ is the group generated by $\{a \otimes b : a \in A,\ b \in B\}$ subject to the relations

$$\begin{aligned}(a + a') \otimes b &= a \otimes b + a' \otimes b, \text{ and}\\ a \otimes (b + b') &= a \otimes b + a \otimes b'.\end{aligned}$$

It is characterized by the following *universal property*: if $f : A \times B \longrightarrow C$ is a bilinear homomorphism (thus $f(a + a', b) = f(a, b) + f(a', b)$ and $f(a, b + b') = f(a, b) + f(a, b')$) then f factors through a unique map $\widetilde{f} : A \otimes B \longrightarrow C$.

Proposition 7.2. *The tensor product satisfies the following properties.*

(i) $A \otimes B \cong B \otimes A$.
(ii) $(\bigoplus_\alpha A_\alpha) \otimes B \cong \bigoplus_\alpha (A_\alpha \otimes B)$.
(iii) $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.
(iv) $\mathbb{Z} \otimes B \cong B$.

(v) $\mathbb{Z}_n \otimes B \cong B/nB$.
(vi) *Given homomorphisms* $f : A \longrightarrow C$ *and* $g : B \longrightarrow C$ *there is a natural homomorphism* $f \otimes g : A \otimes B \longrightarrow C$*. (In particular* $A \otimes -$ *and* $- \otimes B$ *are covariant functors.)*

Remark. The natural setting for the tensor product is for modules M, N over a ring R, generalizing abelian groups which are $\mathbb{Z}$ modules. The module $M \otimes_R N$ is defined as in Definition 7.1 with the additional relation $ra \otimes b = a \otimes rb$ for $r \in R$. It is characterized by an analogous universal property. If M and N are R-algebras (say for commutative R), then $M \otimes_R N$ is naturally an R algebra, and the tensor product can be seen to be the direct product (coproduct) in the category of R-algebras.

8. Tor

As with Hom and Ext, we begin with the failure of exactness for the functor $- \otimes G$.

Proposition 8.1. *For fixed* $G \in \mathsf{AbGp}$ *and any short exact sequence*

$$0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$$

the resulting sequence

$$A \otimes G \longrightarrow A' \otimes G \longrightarrow A'' \otimes G \longrightarrow 0 \tag{18}$$

is exact. In other words, the covariant functor $- \otimes G : \mathsf{AbGp} \longrightarrow \mathsf{AbGp}$ *is only right exact.*

Proof. Exercise. □

The derived functors procedure again gives us a way to extend (18) leftward to a long exact sequence. In this case the groups we obtain, called $\mathrm{Tor}^i(A, G)$ are called the *left derived functors* of $- \otimes G$, since they extend (18) to the left. The combination of right exactness and covariance leads us to use projective groups once again.

Thus, consider the functor $- \otimes G$ for fixed G, and let $A \in \mathsf{AbGp}$ be arbitrary. Let

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \tag{19}$$

be a projective resolution. We apply $- \otimes G$ to the truncated sequence $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$ to obtain the complex

$$\cdots \longrightarrow P_1 \otimes G \longrightarrow P_0 \otimes G \longrightarrow 0 \tag{20}$$

Definition 8.2. The group $\mathrm{Tor}^i(A, G)$ is the ith homology group of (20):

$$\mathrm{Tor}^i(A, G) := \frac{\mathrm{Ker}\,\{P_i \otimes G \longrightarrow P_{i-1} \otimes G\}}{\mathrm{Im}\,\{P_{i+1} \otimes G \longrightarrow P_i \otimes G\}}$$

Recalling the properties of $\mathrm{Ext}^i(A, G)$ proved in Section 4, we have a similar package of results about $\mathrm{Tor}^i(A, G)$.

Proposition 8.3. *The groups* $\mathrm{Tor}^i(A, G)$ *satisfy the following.*
(i) $\mathrm{Tor}^i(A, G)$ *is independent of the choice of projective resolution* (19)*.*
(ii) $\mathrm{Tor}^0(A, G) \cong A \otimes G$.

(iii) *If P is a projective group (in particular if P is free), then* $\mathrm{Tor}^i(P,G) = 0$ *for* $i \geq 1$.
(iv) *For any abelian group A,* $\mathrm{Tor}^i(A,G) = 0$ *for* $i \geq 2$.
(v) *If $0 \to A \to A' \to A'' \to 0$ is a SES, then there is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}^1(A,G) \longrightarrow \mathrm{Tor}^1(A',G) \longrightarrow \mathrm{Tor}^1(A'',G) \\ \longrightarrow A \otimes G \longrightarrow A' \otimes G \longrightarrow A'' \otimes G \longrightarrow 0 \end{aligned} \tag{21}$$

(vi) $\mathrm{Tor}^i(\bigoplus_\alpha A_\alpha, G) = \bigoplus_\alpha \mathrm{Tor}^i(A_\alpha, G)$ *and*

The proof is similar to those in Section (4) and will be left to the reader.

We mention the following analogue of Proposition 5.3.

Proposition 8.4. *The groups* $\mathrm{Tor}^i(A,G)$ *are the same as the left derived functors of $A \otimes -$ (obtained by taking the homology of $A \otimes P_* \longrightarrow 0$ where $P_* \longrightarrow G$ is a projective resolution). In other words,* $\mathrm{Tor}^i(A,G)$ *is symmetric:*

$$\mathrm{Tor}^i(A,G) \cong \mathrm{Tor}^i(G,A).$$

Proof. See [Wei95], Theorem 2.7.2. □

Finally some computations.

Proposition 8.5. $\mathrm{Tor}^0(\mathbb{Z}_m, A) = \mathbb{Z}_m \otimes A = A/mA$ *and* $\mathrm{Tor}^1(\mathbb{Z}_m, A) = {}_mA := \{a \in A : ma = 0\}$.

Proof. Using the resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow 0$$

it follows that $\mathrm{Tor}^i(\mathbb{Z}_m, A)$ are the homology groups of

$$0 \longrightarrow \mathbb{Z} \otimes A \cong A \xrightarrow{m} \mathbb{Z} \otimes A \cong A \longrightarrow 0$$

□

Remark. This hints at the reason for the name Tor, which stands for "Torsion." In fact it is possible to show by taking direct limits (see [Wei95]) that for any $A \in \mathsf{AbGp}$, $\mathrm{Tor}^1(\mathbb{Q}/\mathbb{Z}, A)$ is the torsion subgroup of A.

Definition 8.6. An abelian group G is called *flat* if $- \otimes G$ is exact.

Evidently projective groups are flat (though the converse is not true), and if G is flat then $\mathrm{Tor}^1(A,G) = 0$ for all A. (In fact G is flat *if and only if* $\mathrm{Tor}^1(A,G) = 0$ for all A which is easily seen using symmetry of Tor.) We recall the following fact from algebra.

Proposition 8.7. $\mathbb{Q}$ *is flat. More generally, any localization ring* $S^{-1}\mathbb{Z}$ *is flat.*

Corollary 8.8. $\mathrm{Tor}^1(\mathbb{Q}, A) = 0$ *for all* A.

9. Universal Coefficient Theorem for Homology

We briefly recall the construction of singular homology with coefficients in a group G.

Definition 9.1. The group of singular n-chains on X with G coefficients is the group

$$C_n(X;G) = \left\{ \sum_\alpha n_\alpha \sigma_\alpha \;\middle|\; \sigma_\alpha : \Delta_n \longrightarrow X,\ n_\alpha \in G \right\} \cong C_n(X) \otimes G$$

It is evidently a direct sum over $\{\sigma : \Delta_n \longrightarrow X\}$ of copies of G. It follows that

$$C_n(X,A;G) := C_n(X;G)/C_n(A;G) \cong C_n(X,A) \otimes G.$$

The singular homology groups with coefficients in G are the groups

$$H_n(X;G) := H_n\big(C_*(X) \otimes G\big), \quad H_n(X,A;G) = H_n(C_*(X,A) \otimes G)$$

In analogy to the UCT for cohomology, the Universal Coefficient Theorem for Homology relates the groups $H_n(X;G)$ to the groups $H_n(X) \otimes G$ and $\mathrm{Tor}^1(H_{n-1}(X), G)$. As before, we prove a completely algebraic version first.

Theorem 9.2. *Let C_* be a chain complex of free groups, and let H_n denote its nth homology group. There are short exact sequences*

$$0 \longrightarrow H_n \otimes G \xrightarrow{h} H_n(C_* \otimes G) \xrightarrow{j} \mathrm{Tor}^1(H_{n-1}, G) \longrightarrow 0$$

which are natural in G and C_* and which split (though not naturally with respect to C_*.)

Proof. From the short exact sequence (11) we obtain the sequence

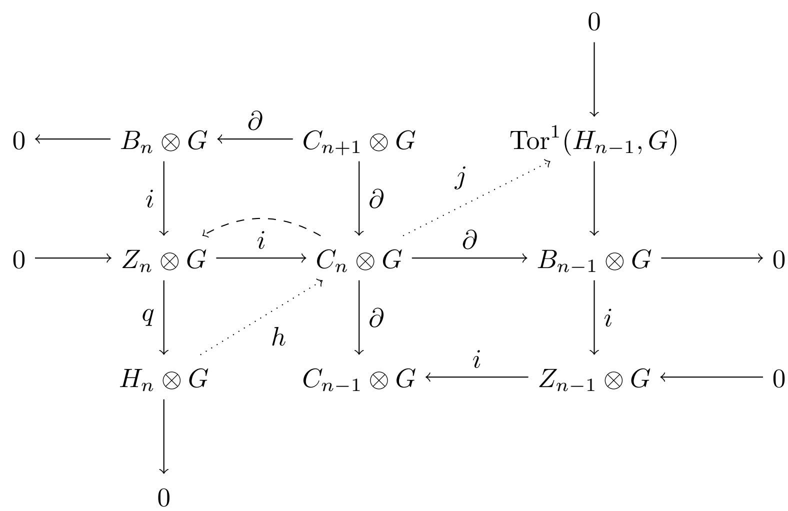
$$0 \longrightarrow Z_n \otimes G \xrightarrow{i\otimes 1} C_n \otimes G \xrightarrow{\partial \otimes 1} B_{n-1} \otimes G \longrightarrow 0$$

which splits, and the (12) gives rise to the sequence

$$0 \longrightarrow \mathrm{Tor}^1(H_n, G) \longrightarrow B_n \otimes G \xrightarrow{i\otimes 1} Z_n \otimes G \xrightarrow{q\otimes 1} H_n \otimes G \longrightarrow 0$$

using the fact that Z_n is free.

The proof follows from a diagram chase analogous to the one in the proof of Theorem 6.1, using the diagram



□

Corollary 9.3 (Universal coefficient theorem for homology)**.** *There are short exact sequences*

$$0 \longrightarrow H_n(X, A) \otimes G \longrightarrow H_n(X, A; G) \longrightarrow \mathrm{Tor}^1\big(H_{n-1}(X, A), G\big) \longrightarrow 0$$

which are natural in G and (X, A) and which split (though not naturally with respect to (X, A).)

References

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Brown University, Department of Mathematics
E-mail address: ckottke@math.brown.edu