ROW REDUCTION AND ITS MANY USES

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These notes will cover the use of row reduction on matrices and its many applications, including solving linear systems, inverting linear operators, and computing the rank, nullspace and range of linear transformations.

1. Linear systems of equations

We begin with linear systems of equations. A general system of m equations in n unknowns $x_1, \ldots, x_n$ has the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_1 \\ \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

where a_{ij} and b_k are valued in a field $\mathbb{F}$, either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

In more compact form, this is equivalent to the equation

$$Ax = b$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad b = (b_1, \ldots, b_m) \in \mathbb{F}^m, \quad x = (x_1, \ldots, x_n) \in \mathbb{F}^n.$$

Finally, observe that the system (1) is completely encoded by the *augmented matrix*

$$[A \mid b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right], \tag{2}$$

where each row of the matrix corresponds to one of the equations in (1).

2. Row reduction and echelon form

The strategy to solving a system of equations such as (1) is to change it, via a series of "moves" which don't alter the solution, to a simpler system of equations where the solution may be read off easily. The allowed moves are as follows:

(1) Multiply an equation on both sides by a nonzero constant $a \in \mathbb{F}$.
(2) Exchange two rows.
(3) Add a scalar multiple of one equation to another.

It is easy to see that each of these is reversible by moves of the same type (multiply the equation by $1/a$, exchange the same two rows again, subtract the scalar multiple of the one equation from the other, respectively), and that after any such move, any solution to the new equation is a solution to the old equation and vice versa. In terms of the augmented matrix $[A \mid b]$, these moves correspond to so-called "row operations."

Definition 1. An *elementary row operation* (or just *row operation* for short) is an operation transforming a matrix to another matrix of the same size, by one of the following three methods:

(1) Multiply row i by $a \in \mathbb{F} \setminus \{0\}$.
(2) Exchange rows i and j.
(3) Add a times row i to row j for some $a \in \mathbb{F}$.

Example 2. The sequence of matrices

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix} 9 & 6 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

is obtained by succesive row operations — first adding -2 times row 1 to row 2, then exchanging rows 1 and 2, and finally multiplying row 1 by 3.

Suppose the matrix in question is $[A \mid b]$, representing the system (1). Using row operations, how should we simplify it so that the solution may be more easily determined? The next definition gives an answer.

Definition 3. A matrix is said to be in *row-echelon form* (or just *echelon form* for short), if

(1) All rows consisting entirely of zeroes are at the bottom,
(2) The first nonzero entry in each row (called the *leading entry* or *pivot*) is equal to 1, and
(3) The pivot in each row lies in a column which is strictly to the right of the pivot in the row above it.

(The third condition means that the pivots must occur in a kind of "descending staircase" pattern from the upper left to the lower right.) The matrix is further said to be in *reduced row-echelon form* if in addition,

(4) Each pivot is the only nonzero entry in its column.

This definition applies generally to any matrix, not just augmented matrices representing systems of equations.

Example 4. The matrix

$$\begin{bmatrix} \boxed{1} & 2 & 4 & 7 & 4 \\ 0 & 0 & \boxed{1} & 9 & 6 \\ 0 & 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in echelon form but not reduced echelon form and boxes have been drawn around the pivot entries. The matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

is in neither echelon nor reduced echelon form. The matrix

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in reduced echelon form.

In the process of *row reduction*, one takes a matrix A and alters it by successive row operations to get a matrix A_e in echelon or A_{re} in reduced echelon form, depending on the application. We will see examples in the next sections.

Note that the echelon form of a matrix obtained by row reduction is not unique, since multiples of the lower rows may always be added to upper ones. However, the reduced echelon form, A_{re} of any matrix is in fact unique, since it encodes a specific linear relationship between the columns of the original matrix, as we shall see.

3. Solving linear systems

To solve a particular linear system by row reduction, start with its augmented matrix and, starting from upper left to lower right, perform succesive row operations to get a 1 in each pivot position and then kill off the entries below it, resulting in a matrix in echelon form.

When a linear system has been reduced to echelon form (or optionally even further to reduced echelon form), the solution (if it exists) may be read off fairly immediately by *back subsitution*.

For example, if we start with the linear system represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 7 & 10 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 6 \end{array}\right]$$

we get an echelon form by

$$\left[\begin{array}{ccc|c} 2 & 4 & 7 & 10 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 10 \\ 1 & 2 & 3 & 6 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This is now equivalent to the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_3 &= 2 \\ 0 &= 1 \end{aligned}$$

which has no solution because of the last equation.

If on the other hand, the echelon form matrix had been

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

corresponding to the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_3 &= 2 \\ 0 &= 0 \end{aligned}$$

then we would conclude that $x_3 = 2$, then back substitute it into the upper equation to get

$$x_1 + 2x_2 = -2.$$

Note that the value of x_2 is not determined — this corresponds to the fact that there is no pivot in the second column of the echelon matrix. Indeed, x_2 is a so-called *free variable*. This means that there are solutions to the equation having any value for $x_2 \in \mathbb{F}$, so the solution is not unique. In any case, once we have chosen a value for x_2, the value for x_1 is determined to be

$$x_1 = -2 - 2x_2.$$

Alternatively, we could have gone further with the row reduction to get the reduced echelon matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

from which we read off the same solution:

$$\begin{aligned} x_1 &= -2 - 2x_2 \\ x_3 &= 2. \end{aligned}$$

The extra work of getting to the reduced echelon form is essentially the same as the work involved in the back substitution step. We summarize our observations in the following.

Proposition 5. *Consider the equation $Ax = b$.*

(1) *If after row reduction, an echelon form for the augmented matrix $[A \mid b]$ has a row of the form*
$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & b \end{array}\right], \qquad b \neq 0$$
then the equation has no solution.

(2) *Otherwise solutions may be found by iteratively solving for the variables from the bottom up and substituting these into the upper equations.*

(3) *Variables whose columns have no pivot entries in the echelon matrix are free, and solutions exist with any values chosen for these variables. If any free variables exist therefore, the solution is not unique.*

An important theoretical consequence of the above is

Theorem 6. *Let $A \in \operatorname{Mat}(m, n, \mathbb{F})$, and let A_e be an echelon form matrix obtained from A by row operations.*

(1) *A solution to $Ax = b$ exists for all $b \in \mathbb{F}^m$ if and only if A_e has pivots in every row.*

(2) *Solutions to $Ax = b$ are unique if and only if A_e has pivots in every column.*

(3) *A is invertible if and only if A_e has pivots in every row and column, or equivalently, if the reduced echelon form A_{re} is the identity matrix.*

Note that the theorem refers to the echelon form for A, rather than for the augmented matrix $[A \mid b]$. The point is that A_e contains information about existence and uniqueness of $Ax = b$ for *all* $b \in \mathbb{F}^m$.

Proof. If A_e fails to have a pivot in row i, this means that row i consists entirely of zeroes. However in this case we can find $b \in \mathbb{F}^m$ such that after row reduction, an echelon form for the augmented matrix $[A \mid b]$ has a row of the form

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & b' \end{array}\right], \qquad b' \neq 0,$$

hence $Ax = b$ has no solution. Conversely if A_e has pivots in all rows, the row reduction of $[A \mid b]$ will never have such a row, and $Ax = b$ will always have at least one solution.

For the second claim, note that having a pivot in every column means there are no free variables, and every variable is completely determined, hence if a solution exists it is unique.

Finally, invertibility of A is equivalent to $Ax = b$ having a unique solution for every b, hence equivalent to A_e having pivots in every row *and* column. The only way this is possible is if A is a square matrix to begin with, and A_e has pivots at each entry along the main diagonal. Clearing out the entries above the pivots, it follows that the reduced echelon form A_{re} for A is the identity matrix. $\square$

4. Elementary matrices

Next we discuss another way to view row reduction, which will lead us to some other applications.

Proposition 7. *Performing a row operation on a matrix is equivalent to multiplying it on the left by an invertible matrix.*

Proof. We exibit the matrices associated to each type of row operation, leaving it as a simple exercise for the reader to verify that multiplication by the matrix corresponds to the given operation.

To multiply row i by $a \in \mathbb{F} \setminus \{0\}$, multiply by the matrix

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & a & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

Here a is in the (i, i) place, and the matrix is otherwise like the identity matrix, with ones along the diagonal and zeroes elsewhere.

To exchange rows i and j, multiply by the matrix

$$\begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 0 & & & & 1 & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & 1 & & & & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix}$$

Here the matrix is like the identity, except with 0 in the (i, i) and (j, j) places, and 1 in the (i, j) and (j, i) places.

To add a times row i to row j, multiply by the matrix

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & a & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

Here the matrix is like the identity, but with a in the (j, i) place.

These are invertible matrices, with inverses corresponding to the inverse row operations. $\square$

Definition 8. The matrices corresponding to the elementary row operations are called *elementary matrices.*

This gives an alternate way to see that row operations don't change the solution to a linear equation. Indeed, given the linear equation

$$Ax = b, \tag{3}$$

performing a sequence of row operations corresponding to elementary matrices $E_1, \dots, E_N$ is equivalent to multiplying both sides by the sequence of matrices:

$$E_N E_{N-1} \cdots E_1 A x = E_N E_{N-1} \cdots E_1 b$$

which has the same solutions as (3) since the E_i are invertible.

5. Finding the inverse of a matrix

The concept of elementary matrices leads to a method for finding the inverse of a square matrix, if it exists.

Proposition 9. *A matrix A is invertible if and only if it may be row reduced to the identity matrix. If $E_1, \ldots, E_N$ are the elementary matrices corresponding to the row operations which reduce A to I, then*

$$A^{-1} = E_N \cdots E_1. \tag{4}$$

Proof. The first claim was proved as part of Theorem 6.

Suppose now that A is reduced to the identity by a sequence row operations, corresponding to elementary matrices $E_1, \ldots, E_N$. By the previous result, this means that

$$E_N E_{N-1} \cdots E_1 A = I$$

Thus the product $E_N \cdots E_1$ is a left inverse for A; since it is invertible it is also a right inverse, so (4) follows. □

Corollary 10. *A matrix A is invertible if and only if it can be written as the product of elementary matrices.*

Proof. The product $E = E_1 \cdots E_N$ of elementary matrices is always invertible, since the elementary matrices themselves are, with inverse $E^{-1} = E_N^{-1} \cdots E_1^{-1}$.

Conversely, if A is invertible, then $A^{-1} = E_N \cdots E_1$ for some elementary matrices by the previous result, and it follows that

$$A = E_1^{-1} \cdots E_N^{-1}.$$

Since the inverses of elementary matrices are also elementary matrices, the claim follows. □

So to compute the inverse of a matrix, we can row reduce it to the identity, keeping track of each step, and then compute the product of the corresponding elementary matrices. However this is quite a bit of work, and there is a clever observation that leads to a much simpler method. Indeed, note that

$$A^{-1} = E_N \cdots E_1 = E_N \cdots E_1 \, I$$

and the latter is equivalent to the matrix obtained by performing the same sequence of row operations on the identity matrix as we did on A.

Thus we can simply perform row reduction on A and, at the same time, do the same sequence of row operations on I. Once A has been transformed into I, I will have been transformed into A^{-1}. The most efficient way to do this is to just perform row reduction on the augmented matrix

$$[A \mid I] = \left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nn} & & & 1 \end{array}\right].$$

Its reduced echelon form will be $[E_N \cdots E_1 A \,| E_N \cdots E_1 I] = [I \mid A^{-1}]$. Schematically,

$$[A \mid I] \xrightarrow{\text{row ops}} [I \mid A^{-1}].$$

Example 11. To compute the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

by row reduction, we compute

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & 8 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & 0 & 4 & -2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/4 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1/3 & -2/3 & 0 \\ 0 & 1 & 1 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/4 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1/3 & -2/3 & 0 \\ 0 & 1 & 0 & 5/6 & 1/3 & -1/4 \\ 0 & 0 & 1 & -1/2 & 0 & 1/4 \end{array}\right]$$

Thus the inverse is

$$\begin{bmatrix} 1/3 & -2/3 & 0 \\ 5/6 & 1/3 & -1/4 \\ -1/2 & 0 & 1/4 \end{bmatrix}$$

as can be verified by matrix multiplication.

6. Spanning and independence

We can use the theory of linear equations to answer questions about spanning and linear independence of vectors in $\mathbb{F}^m$ (and likewise in an abstract vector space V, using a choice of basis to identify V with $\mathbb{F}^m$). Indeed, suppose $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a set of vectors in $\mathbb{F}^m$, and consider the equation

$$x_1 v_1 + x_2 v_2 + \cdots x_n v_n = b, \qquad x_i \in \mathbb{F}, \quad b \in \mathbb{F}^m \tag{5}$$

Recall that:

(1) $\mathcal{B}$ spans $\mathbb{F}^m$ if and only if (5) has a solution for every $b \in \mathbb{F}^m$.
(2) $\mathcal{B}$ is linearly independent if and only if (5) has unique solutions, in particular if it has the unique solution $x_1 = x_2 = \cdots = x_n = 0$ for $b = 0 \in \mathbb{F}^m$.

If we let

$$A = \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{bmatrix} \tag{6}$$

be the $m \times n$ matrix whose columns are given by the components of the vectors $v_1, \ldots, v_n$, then (5) is just the equation $Ax = b$.

We have thus expressed the properties of spanning and linear independence for $\mathcal{B}$ in terms of existence and uniqueness for systems of equations with matrix A, so obtaining the following result as an immediate consequence of Theorem 6.

Proposition 12. *Let $\mathcal{B} = \{v_1, \ldots, v_n\} \subset \mathbb{F}^m$, and let $A \in \mathrm{Mat}(m, n, \mathbb{F})$ be the matrix (6) whose kth column is the components of the v_k for $k = 1, \ldots, n$. Then*

(1) *$\mathcal{B}$ spans $\mathbb{F}^m$ if and only if an echelon form for A has pivots in all rows.*
(2) *$\mathcal{B}$ is linearly independent if and only if an echelon form for A has pivots in all columns.*
(3) *$\mathcal{B}$ is a basis if and only if $m = n$ and an echelon form for A has pivots in every row and column (equivalently, the reduced echelon form for A is the identity matrix)*

Example 13. Consider the set $\{(1,1,1),(1,2,3),(1,4,7)\}$ of vectors in $\mathbb{R}^3$. To determine if they are linearly independent, and hence a basis, we compute

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since an echelon form for A has neither pivots in all rows nor all columns, we conclude that the set is linearly dependent, and does not span $\mathbb{R}^3$.

In fact in the previous example we can go a bit further. The reduced echelon form for A is

$$A_{re} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}$$

and from this it is easy to see that

$$u_3 = -2u_1 + 3u_2,$$

where u_1, u_2 and u_3 are the columns of A_{re}. You can check that the same relation holds between the columns of the original matrix, i.e. our vectors of interest:

$$(1,4,7) = -2(1,1,1) + 3(1,2,3)$$

Similarly, it is immediate that u_1 and u_2 are independent, and so are $(1,1,1)$ and $(1,2,3)$.

What we have noticed is a general phenomenon:

Proposition 14. *Row operations preserve linear relations between columns. That is, if $v_1, \dots, v_n$ are the columns of a matrix A, and $u_1, \dots, u_n$ are columns of the matrix A' which is the result of applying row operations to A, then*

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \iff a_1u_1 + a_2u_2 + \cdots a_nu_n = 0$$

Proof. The fact that A' is the result of applying row operations to A means that $A' = EA$ where $E = E_N \cdots E_1$ is an invertible matrix. So the statement is equivalent to saying that linear relations between columns are left invariant under left multiplication by invertible matrices.

Indeed, one consequence of matrix multiplication is that if $A' = EA$, then the columns of A' are given by E times the columns of A:

$$u_k = Ev_k, \quad \text{for all } k = 1, \dots, n.$$

It follows that

$$\begin{aligned} a_1v_1 + \cdots + a_nv_n = 0 \iff & E(a_1v_1 + \cdots + a_nv_n) \\ &= a_1Ev_1 + \cdots + a_nEv_n \\ &= a_1u_1 + \cdots + a_nu_n = 0. \quad \square \end{aligned}$$

7. Computing nullspaces and ranges

The previous result leads to the final application of row reduction: a method for computing the nullspace and range of a matrix $A \in \mathrm{Mat}(m, n, \mathbb{F})$, viewed as a linear transformation

$$A : \mathbb{F}^n \longrightarrow \mathbb{F}^m.$$

We begin with the range of A. First observe that if we denote the columns of A by $v_1, \ldots, v_n \in \mathbb{F}^m$, as in (6), then

$$\mathrm{Ran}(A) = \mathrm{span}(v_1, \ldots, v_n) \subseteq \mathbb{F}^m.$$

Indeed, by definition $\mathrm{Ran}(A)$ consists of those vectors $b \in \mathbb{F}^m$ such that $b = Ax$ for some $x \in \mathbb{F}^n$, and from the properties of matrix multiplication,

$$\begin{bmatrix} | \\ b \\ | \end{bmatrix} = Ax = x_1 \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + \cdots + x_n \begin{bmatrix} | \\ v_n \\ | \end{bmatrix}$$

Thus $b \in \mathrm{Ran}(A)$ lies in the span of $\{v_1, \ldots, v_n\}$ and vice versa. For this reason the range of A is sometimes referred to as the *column space of* A.

We would like to extract from this spanning set an actual basis for $\mathrm{Ran}(A)$, which we may do as follows.

Proposition 15. *A basis for* $\mathrm{Ran}(A)$ *is given by those columns of* A *which correspond to pivot columns in* A_e*. In other words, if we denote the columns of* A *by* $\{v_1, \ldots, v_n\}$ *and the columns of the echelon matrix* A_e *by* $\{u_1, \ldots, u_n\}$*, then* v_k *is a basis vector for* $\mathrm{Ran}(A)$ *if* u_k *contains a pivot in* A_e*.*

In particular, the rank of A *is given by the number of pivots in* A_e*.*

It is particularly important that we take the columns of the *original* matrix, and not the echelon matrix. While row operations/left multiplication by elementary matrices preserves linear *relations* between columns, they certainly do *not* preserve the span of the columns.

Proof. The pivot columns of A_e are the same as the pivot columns of A_{re}, so consider the reduced echelon form

$$A_{re} = \begin{bmatrix} 1 & * & \cdots & 0 & * & \cdots & 0 & \cdots & * \\ 0 & 0 & \cdots & 1 & * & \cdots & 0 & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \vdots \\ \vdots & & & \vdots & & & \vdots & \cdots & * \\ \vdots & & & \vdots & & & \vdots & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

By inspection, the columns containing pivot entries — the pivot columns — are linearly independent. In addition, any column not containing pivots can be written as a linear combination of the pivot columns (in fact it can be written just as linear combinations of the pivot columns appearing to the left of the column in question, but this is not particularly important here).

By Proposition 14 the same is therefore true of the original columns of A: those which become pivot columns in A_{re} are linearly independent, and the others may be written in terms of them, so they form a basis for $\mathrm{span}(v_1, \ldots, v_n) = \mathrm{Ran}(A)$, as claimed. $\square$

Now on to $\mathrm{Null}(A)$. The process of finding a basis for $\mathrm{Null}(A)$ is really just the process of solving $Ax = 0$, since by definition $\mathrm{Null}(A) = \{x \in \mathbb{F}^n : Ax = 0\}$. Since the right hand side is 0 here, writing down the augmented matrix $[A \mid 0]$ is superfluous and we may just perform row reduction on A itself.

Once an echelon form for A is obtained, say for example

$$A_e = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we translate it back into the linear system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 4x_4 &= 0 \\ x_3 + 2x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

Here the variables x_2 and x_4 are free since they come from non-pivot columns. The trick is now to write down the equation for (x_1, x_2, x_3, x_4) in terms of the free variables, letting these free variables act as arbitrary coefficients on the right hand side. For example, we have

$$\begin{cases} x_1 = 2x_2 - 3x_3 - 4x_4 \\ x_2 = a_2 \text{ free} \\ x_3 = -2x_4 \\ x_4 = a_4 \text{ free} \end{cases} \implies \begin{cases} x_1 = 2a_2 + 2a_4 \\ x_2 = a_2 \\ x_3 = -2a_4 \\ x_4 = a_4 \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

We deduce that $\{(2, 1, 0, 0), (2, 0, -2, 1)\}$ spans $\mathrm{Null}(A)$, and it is also a basis since it is linearly independent. Note that the independence is a consequence of the fact that each vector has a unique coefficient which is 1 where the other vector has a 0. This property in turn comes from the 'free' equations $x_2 = a_2$ and $x_4 = a_4$.

Proposition 16. *To determine a basis for* $\mathrm{Null}(A)$*, the equation* $Ax = 0$ *may be solved by row-reduction. If the general solution to this equation is written using the free variables as coefficients, then the vectors appearing in the linear combination for the solution form a basis for* $\mathrm{Null}(A)$.

In particular, the dimension of $\mathrm{Null}(A)$ *is equal to the number of non-pivot columns of* A_e.

Proof. It is clear that the vectors so obtained span $\mathrm{Null}(A)$, since by definition any solution to $Ax = 0$ may be written as a linear combination of them. That they are linearly independent follows from the equations $x_j = a_j$ for the free variables, as in the example above. $\square$

Note that the rank-nullity theorem is a consequence of the two previous results, since the dimension of the domain of A is the number of columns, the dimension of the nullspace is the number of non-pivot columns, and the dimension of the range is the number of pivot columns.

Example 17. Consider

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 1 & -2 & 4 & 6 \\ 1 & -2 & 2 & 2 \end{bmatrix}$$

An echelon form for A is the matrix from the previous example,

$$A_e = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since columns 1 and 3 contain pivots, it follows that $\{(1,1,1),(3,4,2)\}$ is a basis for $\mathrm{Ran}(A)$, and by the previous computation, $\{(2,1,0,0),(2,0,-2,1)\}$ is a basis for $\mathrm{Null}(A)$. The rank of A is 2.

8. Exercises

Some of the following exercies were taken from Sergei Treil's book *Linear Algebra Done Wrong.*

Problem 1. For what $b \in \mathbb{R}$ does

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ b \end{bmatrix}$$

have a solution? Write the general solution for such b.

Problem 2. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

Problem 3. Determine whether or not the set

$$\{(1,0,1,0),(1,1,0,0),(0,1,0,1),(0,0,1,1)\}$$

is a basis for $\mathbb{R}^4$.

Problem 4. Do the polynomials x^3+2x, x^2+x+1, x^3+5 and x^3+3x-5 span $P_3(\mathbb{R})$?

Problem 5. Compute the rank and find bases for the nullspace and range of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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