FUNCTIONAL ANALYSIS MIDTERM FALL 2016

Problem 1. You showed on a homework set that if $M \subset X$ was a closed subspace of a Banach space X, then

$$\|x + M\| = \inf \{\|x + y\| : y \in M\}$$

is a norm on the quotient space X/M. Here are some further problems:

(a) Show that, for every $\varepsilon > 0$, there exists $x \in X$ with $\|x\| = 1$ such that $\|x + M\| \geq 1 - \varepsilon$. [Hint: For any $x' \in X$, there is some $m \in M$ such that $\|x' + m\| \leq \|x' + M\| + \varepsilon$.]
(b) Deduce from (a) that the quotient map $\pi : X \longrightarrow X/M$, $\pi(x) = x + M$, is a bounded linear operator with $\|\pi\| = 1$.
(c) Prove that X/M is complete. [Hint: Prove that every absolutely convergent series in X/M converges—by a result from class, this is an equivalent characterization of completeness.]

Problem 2. Let X be a Banach space. Prove that a linear functional $f : X \longrightarrow \mathbb{C}$ is bounded if and only if $f^{-1}(\{0\})$ is closed. [Hint: For the "if" direction, use Problem 1.(b)]

Problem 3. Let X be a Banach space and $T \in \mathcal{B}(X, X)$ a bounded linear operator such that $\|I - T\| < 1$, where I denotes the identity operator.

(a) Prove that T is invertible, with inverse given by the **Neumann series**

$$T^{-1} = \sum_{n=1}^{\infty} (I - T)^n.$$

(b) Using the previous result, show that if T has bounded inverse and $\|S - T\| < \left\|T^{-1}\right\|^{-1}$, then S is invertible. Conclude that the set of invertible operators in $\mathcal{B}(X, X)$ is open.

Problem 4. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal sequence in a Hilbert space H. Show that the subspace $\{x \in H : x = \sum a_n e_n\}$ of convergent series is equal to the closure of span $\{e_n\}$.

Problem 5. Take for granted the fact that $L^2\big([0,1]\big) = L^2\big([0,1)\big)$ is a separable Hilbert space (for instance, it has a complete orthonormal basis given by $\{1, \sin(2\pi n x), \cos(2\pi m x) : n, m \in \mathbb{N}\}$). Prove that $L^2(\mathbb{R})$ is separable, by writing $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1)$ and identifying $L^2\big([n, n+1)\big)$, $n \in \mathbb{Z}$ with mutually orthogonal subspaces in $L^2(\mathbb{R})$.

Problem 6. Define the sequence space

$$h^{2,1} = \left\{ x = (x_n) \subset \mathbb{C} : \sum_{n=1}^{\infty} (1 + n^2) \left| x_n \right|^2 < \infty \right\}.$$

(a) Show that
$$\langle x, y\rangle = \sum_{n=1}^{\infty}(1+n^2)x_n\overline{y_n}$$
defines an inner product for which $h^{2,1}$ is a Hilbert space.

(b) Show that $h^{2,1} \subset \ell^2$ and $\|x\|_{\ell^2} \leq \|x\|_{h^{2,1}}$ for all $x \in h^{2,1}$.