FUNCTIONAL ANALYSIS MIDTERM FALL 2016

Problem 1. You showed on a homework set that if $M \subset X$ was a closed subspace of a Banach space X, then

$$\|x + M\| = \inf \{\|x + y\| : y \in M\}$$

is a norm on the quotient space X/M. Here are some further problems:

(a) Show that, for every $\varepsilon > 0$, there exists $x \in X$ with $\|x\| = 1$ such that $\|x + M\| \geq 1 - \varepsilon$. [Hint: For any $x' \in X$, there is some $m \in M$ such that $\|x' + m\| \leq \|x' + M\| + \varepsilon$.]

(b) Deduce from (a) that the quotient map $\pi : X \longrightarrow X/M$, $\pi(x) = x + M$, is a bounded linear operator with $\|\pi\| = 1$.

(c) Prove that X/M is complete. [Hint: Prove that every absolutely convergent series in X/M converges—by a result from class, this is an equivalent characterization of completeness.]

Solution.

(a) Let $x' \notin M$. Then by the definition of infimum, for any $\varepsilon' > 0$ there exists $m \in M$ such that

$$\left\|x' + m\right\| \leq \left\|x' + M\right\| + \varepsilon'.$$

Given $\varepsilon > 0$, choose $\varepsilon' > 0$ such that $\varepsilon' / \|x' + M\| < \varepsilon$. Then with m as above let $x = (x' + m)/\|x' + m\|$. We have $\|x\| = 1$ and, since $x \in x'/\|x' + m\| + M$,

$$\begin{aligned}
\|x + M\| &= \left\|x'/\left\|x' + m\right\| + M\right\| \\
&= \left\|x' + m\right\|^{-1} \left\|x' + M\right\| \\
&\geq 1 - \varepsilon'/\left\|x' + m\right\| \\
&\geq 1 - \varepsilon'/\left\|x' + M\right\| \\
&= 1 - \varepsilon.
\end{aligned}$$

(b) Linearity is straightforward and amounts to the statement that $ax + by + M = a(x + M) + b(y + M)$. To see that π is bounded with unit norm, let $\|x\| = 1$. Then $\|\pi(x)\| = \|x + M\| \leq \|x\| = 1$ since $0 \in M$ (this was already used above). Thus π is bounded and $\|\pi\| \leq 1$. On the other hand, by part (a) $\|\pi\| \geq 1 - \varepsilon$ for all $\varepsilon > 0$, so $\|\pi\| \geq 1$, and therefore equality holds.

(c) Suppose $\sum_{n=1}^{\infty} \|x_n + M\| < \infty$. For each n there exists $y_n \in X$ with $y_n \in x_n + M$ such that $\|y_n\| \leq \|x_n + M\| + 2^{-n}$, by the infimum property. Then $\sum_{n=1}^{\infty} \|y_n\| \leq \sum_{n=1}^{\infty} \|x_n + M\| + 1 < \infty$, so $y = \sum_{n=1}^{\infty} y_n$ converges in X by completeness.

The partial sums $s_k = \sum_{n=1}^{k} y_n$ converge to y in X, and by continuity and linearity of π,

$$\pi(s_k) = \sum_{n=1}^{k} \pi(y_n) = \sum_{n=1}^{k} x_n + M \to \pi(x),$$

so $\sum_{n=1}^{\infty} x_n + M$ converges. Since this was an arbitrary absolutely convergent series, it follows that X/M is complete.

□

Problem 2. Let X be a Banach space. Prove that a linear functional $f : X \longrightarrow \mathbb{C}$ is bounded if and only if $f^{-1}(\{0\})$ is closed. [Hint: For the "if" direction, use Problem 1.(b)]

Solution. If f is bounded then it is continuous, and therefore $f^{-1}(\{0\})$ is closed as $\{0\} \subset \mathbb{C}$ is a closed set.

Conversely, suppose $M = f^{-1}(\{0\})$ is closed. Observe that f factors uniquely as a composition $f = \widetilde{f} \circ \pi$, where $\widetilde{f} : X/M \longrightarrow \mathbb{C}$ is given by $\widetilde{f}(x+M) = f(x)$. Since $f(M) \subset \{0\}$ this is well-defined independent of the chosen representative x of $x+M$. Furthermore $\widetilde{f}$ is injective, since $\widetilde{f}(x+M) = 0$ if and only if $f(x) = 0$, in which case $x \in M$, i.e., $x + M = 0 + M$.

By injectivity of $\widetilde{f}$, $\dim(X/M) \leq \dim(\mathbb{C})$, and therefore $\widetilde{f}$ is automatically bounded, as a linear map on finite-dimensional spaces. By Problem 1.(b), π is bounded, so $f = \widetilde{f} \circ \pi$ is bounded.

The result holds for any linear map $f : X \longrightarrow Y$, provided Y is finite dimensional. If Y is infinite dimensional, then $f^{-1}(\{0\})$ may be closed, yet f unbounded, as for $f : A \subset \ell^\infty \longrightarrow \ell^\infty$, $f((s_n)) = (ns_n)$, where A is the subspace of finitely non-zero sequences, in which example $f^{-1}(\{0\}) = \{0\}$. □

Problem 3. Let X be a Banach space and $T \in \mathcal{B}(X, X)$ a bounded linear operator such that $\|I - T\| < 1$, where I denotes the identity operator.

(a) Prove that T is invertible, with inverse given by the **Neumann series**

$$T^{-1} = \sum_{n=1}^{\infty} (I - T)^n.$$

(b) Using the previous result, show that if T has bounded inverse and $\|S - T\| < \left\|T^{-1}\right\|^{-1}$, then S is invertible. Conclude that the set of invertible operators in $\mathcal{B}(X, X)$ is open.

Solution. Whoops, there was a typo! The series should start at $n = 0$, where $(I - T)^0 := I$.

(a) The series $\sum_{n=0}^{\infty} (I - T)^n$ is absolutely convergent as $\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n < \infty$, the latter being a convergent geometric series. Since $\mathcal{B}(X, X)$ is a Banach space, it follows that $S = \sum_{n=0}^{\infty} (I - T)^n$ converges to a bounded operator. In particular $(I - T)^n \to 0$. Then

$$TS = \big(I - (I - T)\big)S = \lim_k \big(I - (I - T)\big) \sum_{n=0}^{k} (I - T)^n = \lim_k I - (I - T)^{k+1} = I,$$

and similarly $ST = I$. It follows that T is invertible and $T^{-1} = S$.

(b) Suppose T has bounded inverse and $\|S - T\| < \left\|T^{-1}\right\|^{-1}$. Then

$$\left\|I - T^{-1}S\right\| = \left\|T^{-1}(T - S)\right\| \leq \left\|T^{-1}\right\| \|S - T\| < 1,$$

so $T^{-1}S$ is invertible by part (a). It follows that S is invertible with inverse $S^{-1} = (TT^{-1}S)^{-1} = (T^{-1}S)^{-1}T^{-1}$.

If $\mathcal{U}$ denotes the set of invertible operators in $\mathcal{B}(X, X)$, we have just shown that $T \in \mathcal{U}$ implies the open ball $B(T, \left\|T^{-1}\right\|^{-1})$ lies in $\mathcal{U}$, so $\mathcal{U}$ is open.

□

Problem 4. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal sequence in a Hilbert space H. Show that the subspace $\{x \in H : x = \sum a_n e_n\}$ of convergent series is equal to the closure of span $\{e_n\}$.

Solution. If $x = \sum a_n e_n$ then $x = \lim x_k$, where $x_k = \sum_{n=1}^{k} a_n x_n \in \operatorname{span}\{e_n\}$, so it follows that the subspace $\mathcal{C} = \{x \in H : x = \sum a_n e_n\}$ is contained in $\overline{\operatorname{span}\{e_n\}}$.

In the other direction, suppose (x_k) is a sequence in span $\{e_n\}$ that converges in H to $x \in \overline{\operatorname{span}\{e_n\}}$, and let $y = \sum \langle x, e_n \rangle e_n \in \mathcal{C} \subset \overline{\operatorname{span}\{e_n\}}$, which converges by Bessel's inequality. Then $\langle y - x, e_n \rangle = 0$ for all n by orthonormality and we conclude

$$y - x \in \overline{\operatorname{span}\{e_n\}} \cap \overline{\operatorname{span}\{e_n\}}^{\perp} = \{0\},$$

so $x = y \in \mathcal{C}$. □

Problem 5. Take for granted the fact that $L^2([0,1]) = L^2([0,1))$ is a separable Hilbert space (for instance, it has a complete orthonormal basis given by $\{1, \sin(2\pi nx), \cos(2\pi mx) : n, m \in \mathbb{N}\}$). Prove that $L^2(\mathbb{R})$ is separable, by writing $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[n, n+1)$ and identifying $L^2([n, n+1))$, $n \in \mathbb{Z}$ with mutually orthogonal subspaces in $L^2(\mathbb{R})$.

Solution. Extension by zero defines an injective isometry $\varepsilon_n : L^2([n, n+1]) \longrightarrow L^2(\mathbb{R})$ for each n, so we may regard $L^2([n, n+1])$ as a (closed) subspace of $L^2(\mathbb{R})$. Furthermore, elements of $L^2([n, n+1])$ and $L^2([m, m+1])$ for $m \neq n$ are orthogonal in $L^2(\mathbb{R})$, so the subspaces are mutually orthogonal.

Each $L^2([n, n+1])$ has a countable orthonormal basis $\{1_{[n,n+1)}, \sin(2\pi kx)_{[n,n+1)}, \cos(2\pi mx)_{[n,n+1)}\}$ (the subscripts denote multiplication by the characteristic function of $[n, n+1)$), so taking the union of these gives a countable orthonormal set in $L^2(\mathbb{R})$.

It remains to show that this set is total. But if $f \in L^2(\mathbb{R})$ is orthogonal to each of the elements, then it follows by the basis property that $f|_{L^2([n,n+1])} = 0$, which is equivalent to $f = 0$ a.e. on $[n, n+1)$. This holds for each n and hence $f = 0$ (almost) everywhere, so $f = 0$ in $L^2(\mathbb{R})$. □

Problem 6. Define the sequence space

$$h^{2,1} = \left\{ x = (x_n) \subset \mathbb{C} : \sum_{n=1}^{\infty} (1+n^2)\,|x_n|^2 < \infty \right\}.$$

(a) Show that

$$\langle x, y\rangle = \sum_{n=1}^{\infty} (1+n^2) x_n \overline{y_n}$$

defines an inner product for which $h^{2,1}$ is a Hilbert space.

(b) Show that $h^{2,1} \subset \ell^2$ and $\|x\|_{\ell^2} \leq \|x\|_{h^{2,1}}$ for all $x \in h^{2,1}$.

Solution.

(a) That $h^{2,1}$ is a vector space is easy to show (use the triangle inequality). Likewise, the sesquilinearity and nonnegativity of $\langle\cdot,\cdot\rangle$ is straightforward on sequences for which it is defined, and the polarization identity

$$\langle x, y\rangle = \frac{1}{4}\sum_{j=0}^{4} i^j \left\|x + i^j y\right\|_{h^{2,1}}^2$$

shows that $\langle x, y\rangle$ is defined for all $x, y \in h^{2,1}$, where

$$\|x\|_{h^{2,1}}^2 = \sum_{n=1}^{\infty} (1+n^2)\,|x_n|^2$$

is the associated norm, finiteness of which is the defining condition for $h^{2,1}$.

Thus $h^{2,1}$ is an inner product space. To see it is complete, suppose (x^k) is Cauchy. Then (x^k_n) is Cauchy in $\mathbb{C}$ for each fixed n, so $x^k_n \longrightarrow x_n \in \mathbb{C}$. To see that $x = (x_n)$ is in $h^{2,1}$ and that $x_n \longrightarrow x$ in $h^{2,1}$, fix $\varepsilon > 0$ and $N \in \mathbb{N}$, and note that

$$\sum_{n=1}^{N} (1-n^2)\left|x^k_n - x^l_n\right|^2 < \varepsilon$$

for k and l sufficiently large. Using continuity of finite sums, we may take the limit $l \to \infty$ and deduce

$$\sum_{n=1}^{N} (1-n^2)\left|x^k_n - x_n\right|^2 \leq \varepsilon$$

for all N, and then $N \to \infty$ shows that $x^k - x \in h^{2,1}$ with $\left\|x^k - x\right\|^2 \leq \varepsilon$ for all k sufficiently large. Since $x^k \in h^{2,1}$, it follows that $x \in h^{2,1}$ and since ε was arbitrary it follows that $x^k \longrightarrow x$ in $h^{2,1}$.

(b) Since $1 + n^2 \geq 1$ for all n, we obtain the desired inequality

$$\|x\|_{h^{2,1}}^2 = \sum_{n=1}^{\infty} (1 + n^2) |x_n|^2 \geq \sum_{n=1}^{\infty} |x_n|^2 = \|x\|_{\ell^2}^2 ,$$

which also shows that the identity map is an injective bounded linear map $h^{2,1} \longrightarrow \ell^2$.

□