Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													

Math 2321 Final Exam

December 12, 2012

Instructor's name____________________ Your name____________________

Answers from your calculator, without supporting work, are worth zero points.

1) Let $\mathbf{v} = (3, -2, -2)$ and $\mathbf{w} = (-4, 1, 1)$.

a) (2 points) Calculate $\cos\theta$, where θ is the angle between the two vectors v and w.

$$\cos\theta = \frac{\underline{v}\cdot\underline{w}}{|\underline{v}||\underline{w}|} = \frac{-12-2-2}{\sqrt{9+4+4}\sqrt{16+1+1}}$$

$$= \frac{-16}{\sqrt{17}\cdot\sqrt{18}} \approx -0.914659.$$

b) (3 points) Calculate the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$.

$$\text{proj}_{\underline{w}}\underline{v} = \left(\frac{\underline{v}\cdot\underline{w}}{|\underline{w}|}\right)\frac{\underline{w}}{|\underline{w}|} = \left(\frac{\underline{v}\cdot\underline{w}}{|\underline{w}|^2}\right)\underline{w} = \frac{-16}{18}(-4,1,1)$$

$$= -\frac{8}{9}(-4,1,1).$$

c) (4 points) Give a vector equation for the line which passes through the point $(1, -2, 4)$ and is orthogonal to the vectors $\mathbf{v}$ and $\mathbf{w}$. Tangent vector to line: $\underline{v}\times\underline{w}$.

$$\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & -2 \\ -4 & 1 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} -2 & -2 \\ 1 & 1 \end{vmatrix}\underline{i} - \begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix}\underline{j} + \begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix}\underline{k} =$$

$$(-2+2)\underline{i} - (3-8)\underline{j} + (3-8)\underline{k} = (0,5,-5).$$

Vector equation for line:

$$(x,y,z) = (1,-2,4) + t(0,5,-5).$$

2) (2 points each) The volume $V = V(p,T)$ of a specific quantity of a gas is a function of the pressure p and the temperature T. Suppose that V is measured in cubic feet, T is in °F, and p is in lb/in^2. Suppose, further, that $V(24, 500) = 23.69$

a) Thinking physically about the situation, should $\partial V/\partial p$ at $(24, 500)$ be positive or negative? Explain briefly.

Negative. As the pressure goes up, if the temperature is held constant, then the volume should go down.

b) Suppose that you can reliably measure V when p changes by as small an increment as 2 lb/in^2 and/or when T changes by as small an increment as 20°F.

If you're going to take a measurement of V at just one new point (p_1, T_1), where $p_1 \geq 24$ and $T_1 \geq 500$, what should you pick for (p_1, T_1), in order to have the data that you need to obtain a good approximation of $\partial V/\partial p$ at $(24, 500)$? (You are **not** being asked to produce the approximation in this part of the problem; you are just supposed to supply the point (p_1, T_1).)

Let $(p_1, T_1) = (26, 500)$.

c) Assume that $V(p_1, T_1) = 21.86$, where (p_1, T_1) is the point that you supplied above. What approximation do you obtain for $\partial V/\partial p$ at $(p,T) = (24, 500)$?

$$\left.\frac{\partial V}{\partial p}\right|_{(24,500)} \approx \frac{V(26,500) - V(24,500)}{26-24} = \frac{21.86 - 23.69}{2}$$

$$= -0.915 \ \text{ft}^3/\text{lb/in}^2 .$$

d) Assume that $\partial V/\partial T = 0.0255$ ft^3/°F, when $(p,T) = (24, 500)$. Combining this with the data above, what do you obtain for the linearization of the function V at $(p,T) = (24, 500)$?

$$L(p,T) = V(24,500) + \left.\frac{\partial V}{\partial p}\right|_{(24,500)} (p-24) + \left.\frac{\partial V}{\partial T}\right|_{(24,500} (T-500)$$

$$\approx 23.69 - 0.915(p-24) + 0.0255(T-500)$$

ft^3.

3) The pressure P, in atmospheres (atm), produced by oxygen in a bottle, with a piston, is given by

$$P = \frac{nRT}{V - 0.03n} - 1.4\left(\frac{n}{V}\right)^2,$$

where n is the number of moles of gas, T is the temperature in Kelvins (K), V is the volume in liters, and R is the gas constant 0.082 L-atm/mol-K. (Note that the constants 0.03 and 1.4 in the formula are assumed to have the appropriate units.)

$$P = nRT(V-0.03n)^{-1} - 1.4n^2V^{-2}.$$

a) (5 points) Find $\frac{\partial P}{\partial V}$ and $\frac{\partial P}{\partial T}$.

$$\frac{\partial P}{\partial V} = nRT \cdot -(V-0.03n)^{-2} + 2.8n^2V^{-3}. \quad \frac{atm}{L}$$

$$\frac{\partial P}{\partial T} = nR(V-0.03n)^{-1}. \quad \frac{atm}{K}$$

b) (5 points) Suppose that n is held constant at $n = 5$. Also, suppose, when $V = 5$ liters and $T = 300$ K, that V is increasing at a rate of 0.5 liters/s and T is increasing at a rate of 10 K/s. Find the rate of change of P, with respect to time, at this moment.

$$\frac{dP}{dt} = \frac{\partial P}{\partial V}\frac{dV}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt}. \qquad n = 5.$$

When $V = 5$ and $T = 300$:

$$\frac{dP}{dt} = \left[-\frac{nRT}{(V-0.03n)^2} + \frac{2.8n^2}{V^3}\right](0.5) +$$

$$\left[\frac{nR}{V-0.03n}\right](10) =$$

$$\left[-\frac{5\cdot(0.082)(300)}{(5-(0.03)5)^2} + \frac{(2.8)25}{5^3}\right](0.5) +$$

$$\left[\frac{5(0.082)}{5-(0.03)(5)}\right]10 \approx -1.489 \ \frac{atm}{s}.$$

4) Suppose that the xy-plane is being heated, and let $T = T(x, y)$ denote the temperature, in °C, at the point (x, y), where x and y are measured in meters. Suppose that, at the point $(5, 10)$, the temperature decreases at a rate of 0.3°C per meter in the **i** direction, and increases at a rate of 0.4°C per meter in the **j** direction.

a) (2 points) What is the gradient vector of the temperature at $(5, 10)$? (If you cannot do this part, make up an answer, so that you can obtain credit in the parts below.)

$$\vec{\nabla} T(5,10) = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right)\Big|_{(5,10)}$$

$$= (-0.3, 0.4)\ \frac{°C}{m} = \frac{1}{10}(-3,4)\ \frac{°C}{m}.$$

b) (3 points) What is the rate of change of T, with respect to distance, in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$? What is the physical meaning of this number?

$$D_{\underline{u}} T(5,10) = \vec{\nabla} T(5,10) \cdot \underline{u}$$

$$\underline{u} = \frac{\underline{v}}{|\underline{v}|} = \frac{3\underline{i} - 4\underline{j}}{5} = \left(\tfrac{3}{5}, -\tfrac{4}{5}\right).$$

$$= \frac{1}{10}(-3,4) \cdot \frac{1}{5}(3,-4)$$

$$= \frac{1}{50}(-9-16) = -\frac{1}{2}\ \frac{°C}{m}.$$

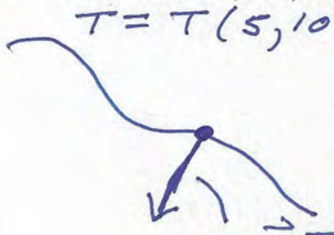
The temperature is dropping at an instan. rate of 0.5 °C per meter in direction $\underline{u}$.

c) (2 points) What is the greatest rate of change of temperature, with respect to distance, at $(5, 10)$?

$$|\vec{\nabla} T(5,10)| = \left|\frac{1}{10}(-3,4)\right| = \frac{1}{10}|(-3,4)| = \frac{1}{2}\ \frac{°C}{m}.$$

d) (2 points) An *isotherm* is a curve along which the temperature is constant. Heat flows along curves which are perpendicular to the isotherms, moving from high temperatures to low temperatures. What is the direction (as a unit vector) in which the heat moves at $(5, 10)$?

$T = T(5,10)$ isotherm



$$\frac{-\vec{\nabla} T(5,10)}{|\vec{\nabla} T(5,10)|} = \frac{-\frac{1}{10}(-3,4)}{\frac{1}{2}} = \frac{1}{5}(3,-4).$$

5) (8 points) Find all critical points of the function $f(x, y) = x^2 + 50y^2 + x^2y$. For each critical point, determine whether it corresponds to a local maximum value, a local minimum value, or a saddle point of f.

$f_x = 2x + 2xy = 0$. $2x(1+y) = 0$. $x = 0$ or $y = -1$.

$f_y = 100y + x^2 = 0$.

$x = 0$. $100y + x^2 = 0$, $y = 0$, $(0,0) = (x,y)$.

$y = -1$. $100y + x^2 = 0$. $x^2 = 100$. $x = \pm 10$. $(x,y) = (10,-1)$ or $(-10,-1)$.

Three crit. pts, $(x,y) = (0,0), (10,-1), (-10,-1)$.

$f_{xx} = 2 + 2y$. $f_{yy} = 100$. $f_{xy} = f_{yx} = 2x$.

$$D = \begin{vmatrix} 2+2y & 2x \\ 2x & 100 \end{vmatrix} = (2+2y)(100) - 4x^2.$$

At $(0,0)$: $D = 200 > 0$. $f_{xx} = 2$. f attains local min.

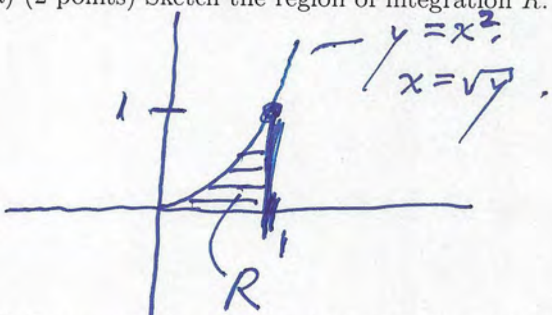
At $(10,-1)$: $D = -400 < 0$. f has a saddle point.

At $(-10,-1)$: $D = -400 < 0$. f has a saddle point.

6) Consider the iterated integral $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy$ and let $\int\int_R \sqrt{x^3+1}\,dA$ be the corresponding double integral.

a) (2 points) Sketch the region of integration R.

$0 \le y \le 1. \quad \sqrt{y} \le x \le 1. \quad \sqrt{y} = x. \quad y = x^2.$



b) (3 points) Give an iterated integral for $\int\int_R \sqrt{x^3+1}\,dA$ in which the order of integration is reversed.

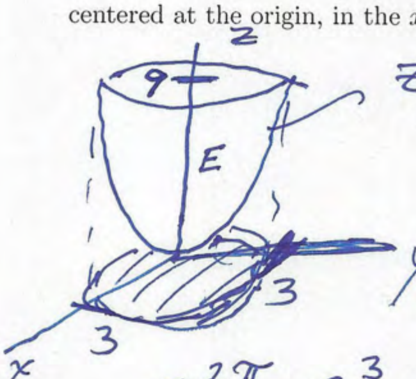
$$\int_0^1 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx.$$

c) (3 points) Evaluate your iterated integral from part b).

$$\int_0^1 \left(\sqrt{x^3+1}\, y \Big|_{y=0}^{y=x^2}\right) dx = \int_0^1 x^2(x^3+1)^{\frac{1}{2}}\,dx = \int_1^2 u^{\frac{1}{2}}\left(\tfrac{1}{3}du\right) = \frac{1}{3}\cdot\frac{u^{3/2}}{3/2}\Big|_1^2 = \frac{2}{9}\left(2^{3/2}-1\right).$$

$\boxed{\text{Let } u = x^3+1.\ du = 3x^2dx.}$

7) (8 points) Consider the "cup" formed by the part of the paraboloid $z = x^2 + y^2$ that lies above the disk of radius 3, centered at the origin, in the xy-plane. If x, y, and z are measured in meters, what volume of liquid can this cup hold?



$$z = x^2+y^2 = r^2. \quad \iiint_E dV =$$

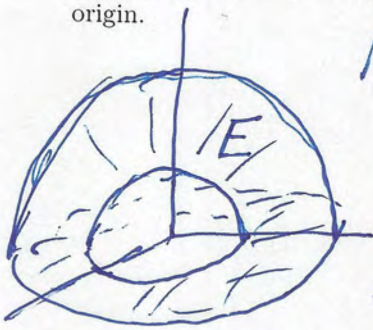
$$\int_0^{2\pi}\int_0^3\int_{r^2}^9 r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^3 r(9-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (9r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{9r^2}{2}-\frac{r^4}{4}\right)\Big|_0^3 d\theta = \int_0^{2\pi}\left(\frac{81}{2}-\frac{81}{4}\right)d\theta =$$

$$\frac{81}{4}(2\pi) = \frac{81\pi}{2}\ m^3.$$

8) (8 points) Find the mass of the region between the upper hemispheres of radius 2 meters and 4 meters, centered at the origin, if the density of the region is given by $\delta(\rho, \phi, \theta) = \rho$ kg/m^3, where ρ is the distance from a point in the region to the origin.



$$\text{Mass} = \iiint_E \delta \, dV =$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_2^4 \rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{\rho^4}{4}\Big|_{\rho=2}^{\rho=4} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{4^4-2^4}{4}\sin\phi \, d\phi \, d\theta$$

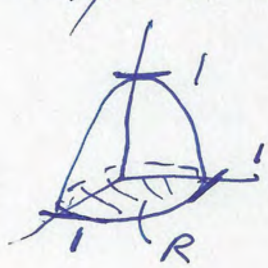
$$= 60\int_0^{2\pi}\left(-\cos\phi\Big|_{\phi=0}^{\phi=\frac{\pi}{2}}\right)d\theta = 60\int_0^{2\pi}(0--1)\,d\theta$$

$$= 120\pi \text{ kg}.$$

9) (8 points) Give an iterated integral for computing the surface area of the part of the paraboloid $z = 1 - x^2 - y^2$, where $z \geq 0$. Your iterated integral must include limits of integration.

DO NOT EVALUATE THIS INTEGRAL.

$$z = f(x,y) = 1 - x^2 - y^2.$$



$$dS = \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx =$$

$$\sqrt{(-2x)^2 + (-2y)^2 + 1}\, dy\, dx.$$

$$\text{Surface area} = \iint_R dS =$$

$$\iint_R \sqrt{4x^2+4y^2+1}\, dy\, dx = \iint_R \sqrt{4r^2+1}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\, r\, dr\, d\theta.$$

10) (8 points) Find the work done by the force field $\mathbf{F} = 2xy\,\mathbf{i} + (x^2+y^2)\,\mathbf{j} = (2xy,\, x^2+y^2)$ Newtons, where x and y are in meters, as it moves a particle from $(4,-2)$ to $(4,2)$ along the curve where $x = y^2$.

At least 3 ways to do this. From definition:

$y = t.\quad x = t^2.\quad \underline{r}(t) = (t^2, t),\quad -2 \le t \le 2.$

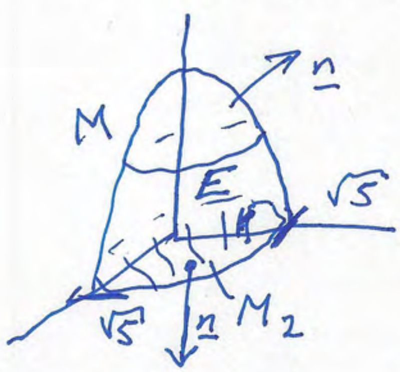
$$\int_C \underline{F}\cdot d\underline{r} = \int_{-2}^{2} \underline{F}(\underline{r}(t))\cdot \underline{r}'(t)\,dt =$$

$$\int_{-2}^{2} (2t^3,\, t^4+t^2)\cdot(2t,\, 1)\,dt = \int_{-2}^{2} (4t^4 + t^4 + t^2)\,dt$$

$$= t^5 + \frac{t^3}{3}\Big|_{-2}^{2} = 32 + \frac{8}{3} - \left(-32 - \frac{8}{3}\right) = 64 + \frac{16}{3}$$

$$= \frac{208}{3} \text{ joules}$$

11) (8 points) Consider the vector field in space $\mathbf{F}(x,y,z) = \left(y\,e^{z^2},\, z\ln(1+x^2),\, -2\right)$. Let M be the portion of the graph $z = 5 - x^2 - y^2$ which sits above the plane $z = 0$, and orient M outwards/upwards. Compute the flux of $\mathbf{F}$ through M.



$$\iint_M \underline{F}\cdot\underline{n}\,dS =$$

Divergence Thm

$$\iint_{M\cup M_2} \underline{F}\cdot\underline{n}\,dS - \iint_{M_2} \underline{F}\cdot\underline{n}\,dS =$$

$$\iiint_E (\vec{\nabla}\cdot\underline{F})\,dV - \iint_{M_2} (y, 0, -2)\cdot(0,0,-1)\,dA$$

$\vec{\nabla}\cdot\underline{F} =$ $0+0+0=0.$

$$= 0 - \iint_{M_2} 2\,dA = -2\left(\text{area of } M_2\right)$$

$$= -2\pi(\sqrt{5})^2 = -10\pi.$$

12) In Maxwell's theory of electrodynamics, the magnetic field $\mathbf{B}(x, y, z)$ throughout space is given by the curl of the "vector potential" $\mathbf{A}(x, y, z)$, i.e., $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$.

Suppose that $\mathbf{A}(x, y, z) = \left(\sin z,\ 2x - z^2,\ x\,e^y\right)$.

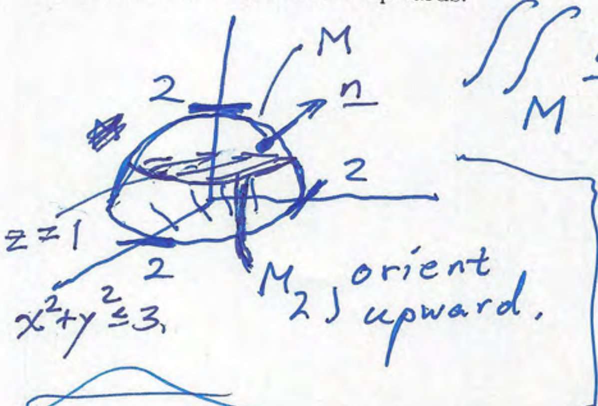
a) (3 points) Compute the magnetic field $\mathbf{B}(x, y, z)$.

$$\vec{\nabla} \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & 2x - z^2 & xe^y \end{vmatrix} =$$

$$\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - z^2 & xe^y \end{vmatrix} \underline{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \sin z & xe^y \end{vmatrix} \underline{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \sin z & 2x - z^2 \end{vmatrix} \underline{k}$$

$$= \left(xe^y + 2z,\ -e^y + \cos z,\ 2\right).$$

b) (5 points) Let M be the portion of the sphere $x^2 + y^2 + z^2 = 4$ which sits above the plane $z = 1$. Compute the flux of $\mathbf{B}$ through M, oriented upwards.



$$\iint_M \underline{B} \cdot \underline{n}\, dS = \boxed{\text{Stokes' Thm.}}$$

$$\iint_M (\vec{\nabla} \times \underline{A}) \cdot \underline{n}\, dS =$$

$$\int_{\partial M} \underline{A} \cdot d\underline{r} = \iint_{M_2} (\vec{\nabla} \times \underline{A}) \cdot \underline{n}\, dS$$

$$= \iint_{M_2} \underline{B} \cdot \underline{n}\, dS = \iint_{M_2} (*, *, 2) \cdot (0, 0, 1)\, dA$$

$$= \iint_{M_2} 2\, dA = 2 \left(\begin{array}{c}\text{area of} \\ M_2\end{array}\right) = 2\pi (\sqrt{3})^2$$

$$= 6\pi.$$