Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													

Math 2321 Final Exam

December 12, 2014

Instructor's name____________________ Your name____________________

Please check that you have 10 different pages.

Answers from your calculator, without supporting work, are worth zero points.

1) (3 points each) Consider the surface M in $\mathbb{R}^3$ where $z = x^2 + y^2$.

a) Find an equation for the tangent plane to the surface at the point $(2, 1, 5)$.

$F(x,y,z) = x^2 + y^2 - z = 0.$ $\vec{\nabla} F = (2x, 2y, -1).$

$\vec{\nabla} F(2,1,5) = (4, 2, -1).$

$\boxed{4(x-2) + 2(y-1) - 1(z-5) = 0.}$ or

or

$\boxed{z = 5 + 4(x-2) + 2(y-1).}$ $\boxed{4x + 2y - z - 5 = 0.}$

b) Give a vector equation, or parametric equations, for the line which passes through the point $(2, 1, 5)$ and is normal to the surface M at $(2, 1, 5)$.

$\boxed{(x, y, z) = (2, 1, 5) + t(4, 2, -1).}$

or

$\boxed{x = 2 + 4t. \quad y = 1 + 2t. \quad z = 5 - t.}$

c) Determine whether or not the point $(1, -3, 2)$ is on your line from part b). You must show your supporting work.

Is there a t such that

$\underbrace{1 = 2 + 4t}_{t = -1/4}$, $\underbrace{-3 = 1 + 2t}_{t = -2}$, and $\underbrace{2 = 5 - t}_{t = 3}$?

$t = -\frac{1}{4} \neq t = -2.$

No.

$\boxed{(1, -3, 2) \text{ is } \underline{\text{not}} \text{ on the line.}}$

2) (4 points each) The wind-chill index $W = W(T, v)$ is the perceived temperature when the actual temperature is T and the wind speed is v. Suppose that W and T are measured in °C and v is measured in km/h. Assume that $W(-20, 50) = -35$ and $W(-20.5, 50) = -35.6$,

a) Use the data to estimate $\partial W/\partial T$ at the point $(T, v) = (-20, 50)$.

$$\left.\frac{\partial W}{\partial T}\right|_{(-20,50)} \approx \frac{W(-20.5,50)-W(-20,50)}{-20.5-(-20)} = \frac{-35.6+35}{-20.5+20}$$

$$= \frac{-0.6}{-0.5} = \boxed{\frac{6}{5} = 1.2 \ \frac{°C}{°C} \text{ (or no units.)}}$$

b) Consider $\partial W/\partial v$ at the point $(T, v) = (-20, 50)$. Intuitively/physically, should this be positive, negative, or zero? Explain.

Negative — if the temperature is fixed, and the wind speed goes up, then the perceived temperature goes down.

3) (9 points) Suppose that $R = R(u, v, w)$ and that $\vec{\nabla}R(3, 1, 2) = \frac{1}{7}(3, 1, 2)$. Also suppose that $u = x + 2y$, $v = 2x - y$, and $w = 2xy$. Find R_x and R_y when $x = y = 1$.

$$R_x = \frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x} = \vec{\nabla}R(u,v,w)\cdot\frac{\partial}{\partial x}(u,v,w).$$

Note that, when $x=y=1$, $u=3$, $v=1$, and $w=2$.

When $x=y=1$: $\boxed{R_x = \frac{1}{7}(3,1,2)\cdot(1,2,2) = \frac{1}{7}(3+2+4) = \boxed{9/7.}}$

$2y$ at $y=1$.

$2x$

$$\boxed{R_y = \frac{1}{7}(3,1,2)\cdot(2,-1,2) = \frac{1}{7}(6-1+4) = 9/7.}$$

4) (9 points) Suppose that electric charge is distributed on a metal plate in such a way that the charge in coulombs, at a point (x, y), in meters, is given by $Q(x, y) = \sin(xy)$. At the point $(3, 0)$, find the maximum rate of change (in coulombs/meter) of the charge and the direction in which it occurs.

$$\vec{\nabla}Q = (y\cos(xy),\ x\cos(xy)).\ \frac{\text{Coulombs}}{m}.$$

$$\vec{\nabla}Q(3,0) = (0,\ 3\cdot 1) = (0,3)\ \frac{\text{Coulombs}}{m}.$$

$$\boxed{\text{Max. rate of change} = |\vec{\nabla}Q(3,0)| = 3 \text{ Coulombs}/m.}$$

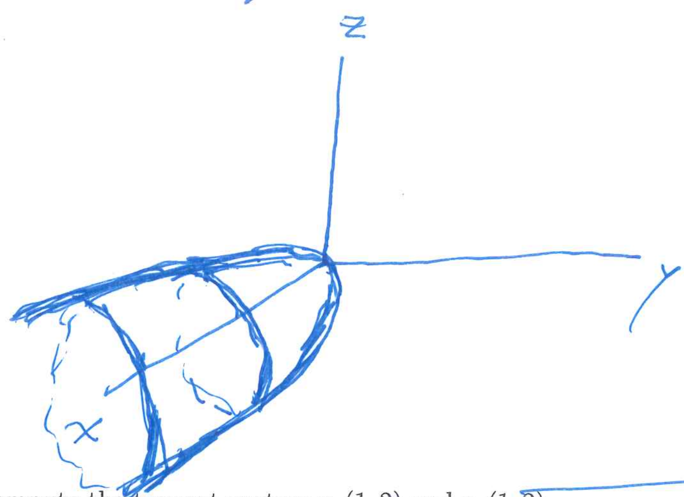
$$\boxed{\text{Direction} = \frac{(0,3)}{|(0,3)|} = (0,1) = \hat{j}}$$

5) (3 points each) Let $\mathbf{r}(u,\ v) = (u^2 + v^2,\ u, v)$ be a parametrization of a surface S in $\mathbb{R}^3$.

a) Give an equation for S in the form $f(x,\ y,\ z) = 0$ (i.e., describe S as a level surface) and sketch the surface S.

$x = u^2 + v^2$. $y = u$. $z = v$. So, $x = y^2 + z^2$, i.e.,

$$\boxed{x - y^2 - z^2 = 0.}$$



b) Compute the tangent vectors $\mathbf{r}_u(1, 2)$ and $\mathbf{r}_v(1, 2)$.

$\underline{r}_u = (2u, 1, 0)$. $\boxed{\underline{r}_u(1,2) = (2, 1, 0).}$

$\underline{r}_v = (2v, 0, 1)$. $\boxed{\underline{r}_v(1,2) = (4, 0, 1).}$

c) Give a parameterization of the tangent plane of S at $\mathbf{r}(1, 2)$.

$(x, y, z) = \underline{r}(1,2) + a\underline{r}_u(1,2) + b\underline{r}_v(1,2).$

$$\boxed{(x, y, z) = (5, 1, 2) + a(2, 1, 0) + b(4, 0, 1).}$$

6) (10 points) Use Lagrange multipliers to find the five critical points of the function $f(x, y, z) = xz - y^2$ restricted to the paraboloid where $\underbrace{x^2 + y + z^2}_{g} = 1$ (i.e., subject to the constraint $x^2 + y + z^2 = 1$).

$x^2 + y + z^2 = 1.$

$\vec{\nabla} f = \lambda \vec{\nabla} g.$ $(z, -2y, x) = \lambda(2x, 1, 2z).$

Solve for (x, y, z):

$x^2 + y + z^2 = 1.$ $z = 2\lambda x.$ $-2y = \lambda.$ $x = 2\lambda z.$

$z = 2\lambda(2\lambda z).$ $\underbrace{z = 4\lambda^2 z.}$

$z = 0$ or $1 = 4\lambda^2$, i.e., $\lambda = \pm \frac{1}{2}.$

Case 1:

$z = 0.$

$x^2 + y = 1. \rightarrow y = 1,$

$-2y = \lambda.$ $x = 0.$

$\boxed{(0, 1, 0),}$

Case 2:

$\lambda = \frac{1}{2}.$

$z = x.$

$-2y = \frac{1}{2}.$ $y = -\frac{1}{4}.$

$x^2 - \frac{1}{4} + x^2 = 1.$

$2x^2 = \frac{5}{4}.$

$x^2 = \frac{5}{8}.$

$x = \pm\sqrt{\frac{5}{8}}.$

$\boxed{\left(\sqrt{\frac{5}{8}}, -\frac{1}{4}, \sqrt{\frac{5}{8}}\right)}$

$\boxed{\left(-\sqrt{\frac{5}{8}}, -\frac{1}{4}, -\sqrt{\frac{5}{8}}\right)}$

Case 3

$\lambda = -\frac{1}{2}.$

$z = -x.$

$-2y = -\frac{1}{2}.$

$y = \frac{1}{4}.$

$2x^2 + \frac{1}{4} = 1.$

$2x^2 = \frac{3}{4}.$

$x = \pm\sqrt{\frac{3}{8}}.$

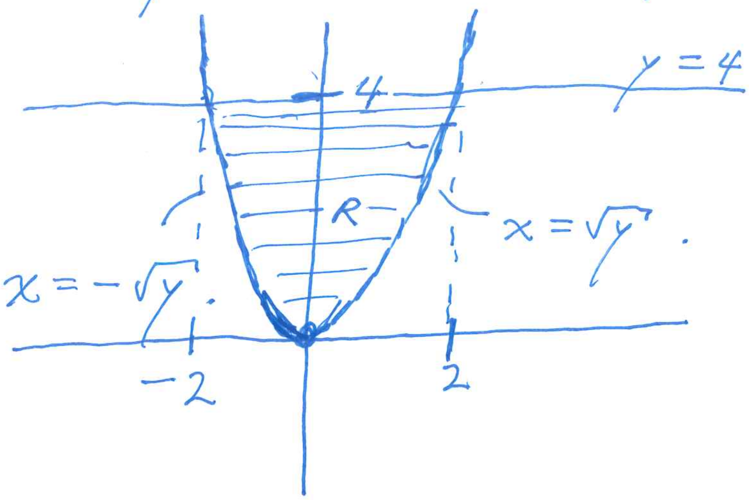
$\boxed{\left(\sqrt{\frac{3}{8}}, \frac{1}{4}, -\sqrt{\frac{3}{8}}\right).}$

$\boxed{\left(-\sqrt{\frac{3}{8}}, \frac{1}{4}, \sqrt{\frac{3}{8}}\right)}$

7) Consider the iterated integral $\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \sqrt{y}\, dxdy$ and let $\int\int_R \sqrt{y}\, dA$ be the corresponding double integral.

a) (4 points) Sketch the region of integration R..

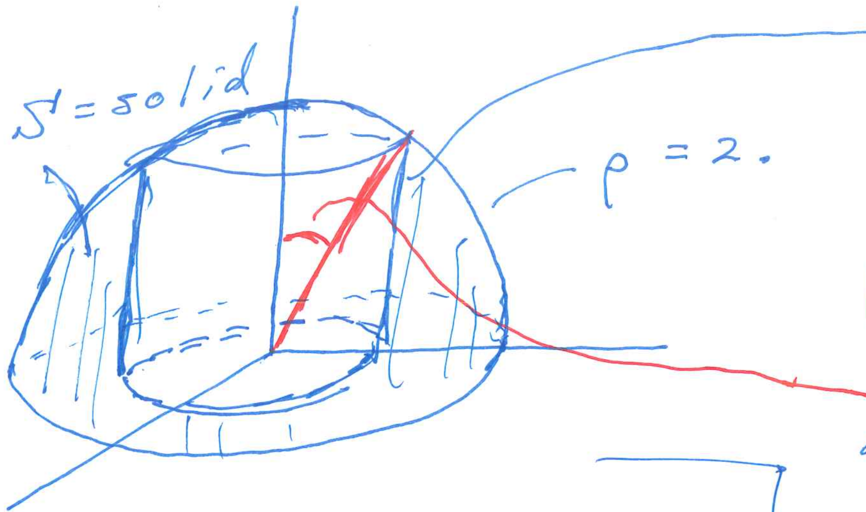
$0 \le y \le 4. \quad -\sqrt{y} \le x \le \sqrt{y}.$



b) (5 points) Give an iterated integral for $\int\int_R \sqrt{y}\, dA$ in which the order of integration is reversed. **Do not evaluate this integral.**

$$\int_{-2}^{2}\int_{x^2}^{4} \sqrt{y}\, dy\, dx.$$

8) (9 points) Consider a solid that occupies the region in $\mathbb{R}^3$ that lies inside the (top) hemisphere where $x^2+y^2+z^2=4$ and $z \geq 0$ but lies outside the cylinder where $x^2+y^2=3$. Assume that x, y, and z are measured in meters. Suppose its density is given by $\delta(x,y,z) = \frac{1}{x^2+y^2+z^2}$ kg/m^3. Find the total mass of the solid.



$$x^2+y^2=3.$$
$$r^2 = 3.$$
$$\rho^2 \sin^2\phi = 3.$$
$$\rho = \frac{\sqrt{3}}{\sin\phi}.$$

ϕ_0, where $\rho = 2$ hits $\rho = \frac{\sqrt{3}}{\sin\phi}$.

$$2\sin\phi_0 = \sqrt{3}.$$
$$\phi_0 = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

$$\text{Mass} = \iiint_{S'} \delta \, dV =$$

$$\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_{\frac{\sqrt{3}}{\sin\phi}}^{2} \frac{1}{\rho^2}\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$\int_0^{2\pi}\int_{\pi/3}^{\pi/2} (\sin\phi)\left(2 - \frac{\sqrt{3}}{\sin\phi}\right) d\phi \, d\theta =$$

$$\int_0^{2\pi}\int_{\pi/3}^{\pi/2} (2\sin\phi - \sqrt{3}) \, d\phi \, d\theta =$$

$$2\pi\left(-2\cos\phi - \sqrt{3}\phi\right)\Big|_{\pi/3}^{\pi/2} = 2\pi\left[0 - \sqrt{3}\frac{\pi}{2} + 2\cdot\frac{1}{2} + \frac{\sqrt{3}\pi}{3}\right]$$

$$= 2\pi\left(1 - \frac{\sqrt{3}\pi}{6}\right) \approx 0.585 \text{ kg.}$$

$P \qquad Q$

9) (7 points) A particle travels through the force field $\mathbf{F} = (3+ye^{xy}, xe^{xy}+2y\sin(y^2))$ Newtons along the following oriented path C: the line segment from the origin to the point (0, 1), then along the line segment from (0, 1) to the point $(2, -1)$, then along the line segment from $(2, -1)$ to the point $(2, 1)$, then along the line segment from $(2, 1)$ to the point $(4, -1)$, and finally, along the line segment from $(4, -1)$ to the point $(4, 0)$. Assume that x and y are in meters. Calculate the work done by $\mathbf{F}$ along C.

(Hint: This can be done **without** parameterizing five different line segments.)

Could check ~~$\vec{\nabla} \times \underline{F}$~~ $Q_x - P_y$ to see if 0 (on simply-conn. region), which would imply $\underline{F}$ is conservative.

$$Q_x = xe^{xy}y + e^{xy} + 0. \quad P_y = ye^{xy}x + e^{xy}. \quad \Big\} \quad Q_x - P_y = 0.$$

$\underline{F}$ is conservatie.

Find potential function $f(x,y)$ so that $\underline{F} = \vec{\nabla} f$, i.e.,

$$\left(3+ye^{xy},\ xe^{xy}+2y\sin(y^2)\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

$$f = \int (3+ye^{xy})\,dx = 3x + e^{xy} + A(y).$$

Find $A(y)$. $\quad \dfrac{\partial f}{\partial y} = 0 + e^{xy}x + A'(y)$

$$= xe^{xy} + 2y\sin(y^2).$$

Need: $A'(y) = 2y\sin(y^2)$. $A(y) = -\cos(y^2) + C$,

$f = 3x + e^{xy} - \cos(y^2)$ (picked $C=0$).

$$\underbrace{\int_C \underline{F}\cdot d\underline{r}}_{\text{Work}} = f(4,0) - f(0,0)$$

$$= 12 + 1 - 1 - (0+1-1) = 12 \text{ joules.}$$

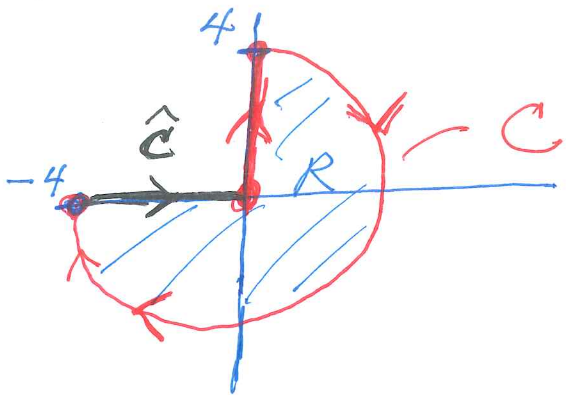
Fundamental Theorem of Line Integrals.

10) (7 points) A particle starts at the origin, moves along a straight line to the point $(0,4)$, then moves clockwise along the circle of radius 4, centered at the origin, to the point $(-4,0)$, where it stops; let C denote this oriented path of the particle.

While traveling, the particle moves through the force field

$$\mathbf{F} = (\underbrace{8xy + 4x - 6y}_{P},\ \underbrace{y^{3y+1} + \sin(\sqrt{y} + y) + 4x^2}_{Q}),$$

where $\mathbf{F}$ is in Newtons, and x and y are in meters. Calculate the work done by F along C, i.e., calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.



$$Q_x - P_y =$$

$$8x - (8x - 6) = 6.$$

$$\int_C \underline{F} \cdot d\underline{r} = \underbrace{\int_{C+\hat{C}} \underline{F} \cdot d\underline{r}}_{\text{use Green's Thm,}} - \int_{\hat{C}} \underline{F} \cdot d\underline{r}$$

oriented clockwise

$$= -\iint_R (Q_x - P_y)\, dA - \int_{\hat{C}} \overbrace{P}^{= 0 + 4x - 0 \text{ along } \hat{C}} dx + Q \underbrace{\cancel{dy}}_{= 0 \text{ along } \hat{C}}$$

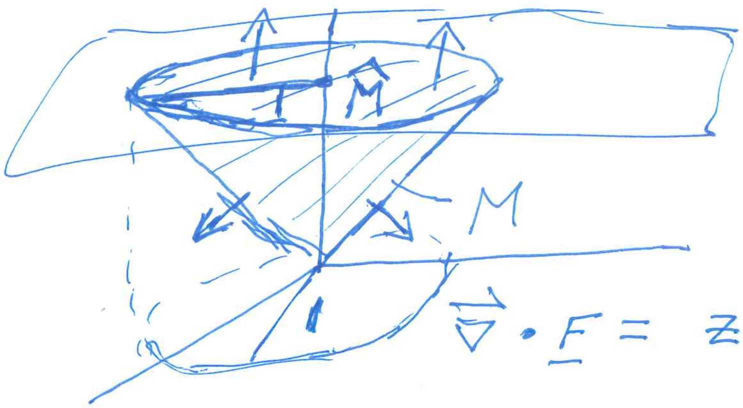
$$= -\iint_R 6\, dA - \int_{-4}^{0} 4x\, dx =$$

$$-6\left(\frac{3}{4}\right)\pi(4)^2 - \left(2x^2\Big|_{-4}^{0}\right)$$

$$-72\pi - (0 - 32) = \boxed{-72\pi + 32 \text{ joules.}}$$

11) (7 points) Consider the vector field $\mathbf{F}(x, y, z) = (xz, yz, z^2)$. Let M be the surface of the half-cone $z = \sqrt{x^2+y^2}$ where $z \le 1$, oriented downward. Calculate the flux integral $\iint_M \mathbf{F}\cdot\mathbf{n}\, dS$ of $\mathbf{F}$ through M.

$\boxed{z = r.}$



Let E = solid enclosed by $M \cup \hat{M}$.

$\vec{\nabla}\cdot \underline{F} = z + z + 2z = 4z.$

$$\iint_M \underline{F}\cdot\underline{n}\, dS = \underbrace{\iint_{M\cup\hat{M}} \underline{F}\cdot\underline{n}\, dS}_{\text{Use Divergence Thm,}} - \iint_{\hat{M}} \underline{F}\cdot\underline{n}\, dS$$

$$= \iiint_E 4z\, dV - \iint_{\hat{M}} (*, *, 1)\cdot(0,0,1)\, dA$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 4zr\, dz\, dr\, d\theta - \iint_{\hat{M}} 1\, dA$$

$$= \int_0^{2\pi}\int_0^1 \left(2z^2 r\Big|_{z=r}^{z=1}\right) dr\, d\theta - \pi(1)^2$$

$$= \int_0^{2\pi}\int_0^1 (2r - 2r^3)\, dr\, d\theta - \pi =$$

$$= \int_0^{2\pi} \left(r^2 - \frac{r^4}{2}\right)\Big|_0^1 d\theta - \pi = \int_0^{2\pi} \frac{1}{2}\, d\theta - \pi$$

$$\boxed{= \pi - \pi = 0.}$$

or can parameterize!

11) (7 points) Consider the vector field $\mathbf{F}(x, y, z) = (xz, yz, z^2)$. Let M be the surface of the half-cone $z = \sqrt{x^2+y^2}$ where $z \le 1$, oriented downward. Calculate the flux integral $\iint_M \mathbf{F}\cdot\mathbf{n}\, dS$ of $\mathbf{F}$ through M.

$$\left.\begin{array}{l} \underline{r}(u,v) = (v\cos u,\ v\sin u,\ v), \\ 0 \le u \le 2\pi.\ \ 0 \le v \le 1, \end{array}\right\} \text{Parameterization of } M.$$

$$\underline{r}_u = (-v\sin u,\ v\cos u,\ 0).$$

$$\underline{r}_v = (\cos u,\ \sin u,\ 1).$$

$0 \le v \le 1$

oriented downward

$$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} = (v\cos u,\ v\sin u,\ -v).$$

$$\iint_M \underline{F}\cdot\underline{n}\, dS = \int_0^1\int_0^{2\pi} \underline{F}(\underline{r}(u,v))\cdot(\underline{r}_u \times \underline{r}_v)\, du\, dv$$

$$\int_0^1\int_0^{2\pi} (v^2\cos u,\ v^2\sin u,\ v^2)\cdot(v\cos u,\ v\sin u,\ -v)\, du\, dv$$

$$= \int_0^1\int_0^{2\pi} (v^3\cos^2 u + v^3\sin^2 u - v^3)\, du\, dv$$

$$= \int_0^1\int_0^{2\pi} 0\, du\, dv = 0.$$

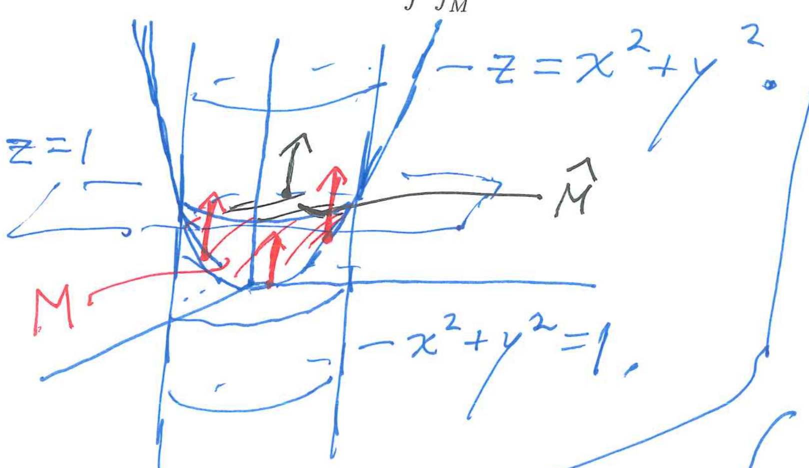
12) Consider the vector field $\mathbf{F}(x, y, z) = (x^2z^2, y^2z^2, xyz)$.

a) (2 points) Calculate the curl, $\vec{\nabla} \times \mathbf{F}$, of $\mathbf{F}$.

$$\vec{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z^2 & y^2z^2 & xyz \end{vmatrix} =$$

$$(xz - 2y^2z, -yz + 2x^2z, 0 - 0).$$

b) (5 points) Let M be the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 1$, oriented upward. Compute the flux of the curl: $\int\int_M (\vec{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\, dS$.



$$\iint_M (\vec{\nabla} \times \underline{F}) \cdot \underline{n}\, dS$$

$$\parallel$$

$$\iint_{\hat{M}} (\vec{\nabla} \times \underline{F}) \cdot \underline{n}\, dS =$$

$$\iint_{\hat{M}} (*, *, 0) \cdot (0, 0, 1)\, dA$$

$$\boxed{= \iint_{\hat{M}} 0\, dA = 0.}$$