Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	total
points										

Calculus 3, Final Exam

April 24, 2014

Instructor's name ______________ Your name ______________

Show all your work in the space provided. Use the back page if necessary. No credit for unjustified answers. You may use a calculator to check your answers but must do all calculations by hand. Formula sheet is allowed.

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(1) (10 points) Suppose that the temperature measured in degrees Celsius at each point of a metal plate is given by $T(x,y) = e^x \cos y + e^y \cos x$, where x and y are given in meters.

a) (4 points) In what direction does the temperature increase most rapidly at the point $(0,0)$?

$\nabla T(x,y) = (e^x \cos y - e^y \operatorname{sen} x \,,\, e(-\operatorname{sen} y) + e^y \cos x)$

$\nabla T(0,0) = (1,1)$

$\|\nabla T(0,0)\| = \sqrt{2}$

The direction in which the temperature increases most rapidly at $(0,0)$ is:

ANSWER: $\frac{\nabla T(0,0)}{\|\nabla T(0,0)\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

b) (3 points) What is the rate of increase in that direction?

The rate of increase is $\|\nabla T(0,0)\| = \sqrt{2}$ °C/m.

ANSWER: $\sqrt{2}$

c) (3 points) In what direction does the temperature decrease most rapidly at the point $(0,0)$?

The direction in which the temperature decreases most rapidly is the opposite of the gradient.

ANSWER: $-\frac{\nabla T(0,0)}{\|\nabla T(0,0)\|} = -\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(2) (10 points) Find the critical points of the function

$$f(x,y) = 2xy^2 - 8y^2 - x^2$$

and classify them as local maxima, minima or saddle points.

critical points: $f_x = f_y = 0$

Ⓐ $f_x(x,y) = 2y^2 - 2x = 0$

Ⓑ $f_y(x,y) = 4xy - 16y = 4(x-4)y = 0 \Rightarrow x=4$ or $y=0$

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$2y^2 - 8 = 0$ $\quad\quad$ $-2x = 0$

$y^2 = 4$ $\quad\quad$ $x = 0$

$y=2$ or $y=-2$

critical points: $(0,0)$, $(4,2)$, $(4,-2)$.

2nd derivative test:

$f_{xx}(x,y) = -2$

$f_{xy}(x,y) = 4y$

$f_{yy}(x,y) = 4x - 16$

$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = -8x + 32 - 16y^2$

$D(0,0) = 32 > 0 \rightarrow$ max or min.

$f_{xx}(0,0) = -2 < 0 \Rightarrow (0,0)$ is local max.

$D(4,2) = -32 + 32 - 16\cdot 4 = -64 < 0 \rightarrow$ saddle

$D(4,-2) = -32 + 32 - 16\cdot 4 = -64 < 0 \rightarrow$ saddle

ANSWER: ____________________

$(0,0)$ - local maximum

$(4,2)$ - saddle

$(4,-2)$ - saddle.

(3) (12 points) Use Lagrange multipliers to find the critical points of the restriction of

$$f(x,y,z) = x + 2y + 3z - 12$$

to the surface given by $x^2 + 2y^2 + 3z^2 = 1$.

$f(x,y,z) = x + 2y + 3z - 12, \Rightarrow \vec{\nabla} f(x,y,z) = (1, 2, 3)$

Constraint $g(x,y,z) = x^2 + 2y^2 + 3z^2 = 1 \Rightarrow \vec{\nabla} g(x,y,z) = (2x, 4y, 6z)$

By Lagrange multipliers, $\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$

$\Rightarrow \begin{cases} 1 = 2\lambda x \\ 2 = 4\lambda y \\ 3 = 6\lambda z \end{cases} \Rightarrow \lambda x = \lambda y = \lambda z = \frac{1}{2}$

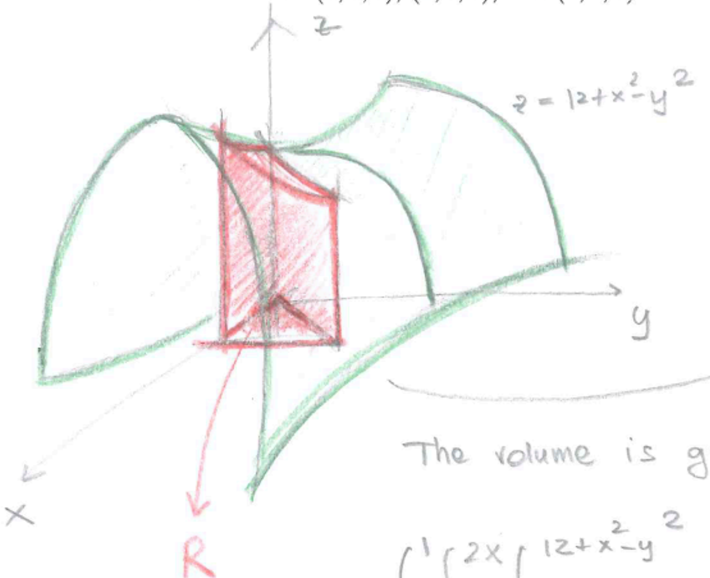
Since $\lambda \neq 0 \Rightarrow x = y = z$ $(*)$

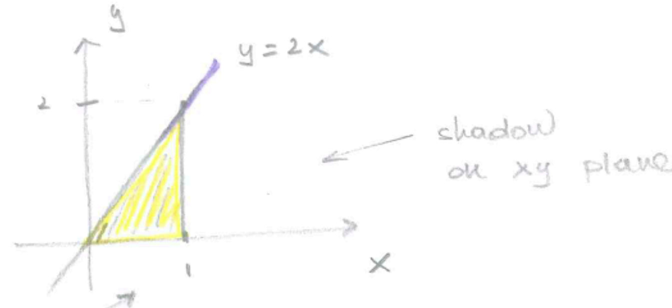
Plug $(*)$ into the constraint, $x^2 + 2y^2 + 3z^2 = 6x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{6}}$

Hence the critical points are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

ANSWER: $\pm\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

(4) (10 points) Calculate the volume below the graph of $z = 12 + x^2 - y^2$ and above the filled-in triangle in the xy-plane which has vertices $(0,0,0)$, $(1,0,0)$, and $(1,2,0)$.





The volume is given by $\int_R dV =$

$$= \int_0^1 \int_0^{2x} \int_0^{12+x^2-y^2} dz\,dy\,dx$$

$$= \int_0^1 \int_0^{2x} (12 + x^2 - y^2)\,dy\,dx$$

$$= \int_0^1 \left(12y + x^2 y - \frac{1}{3}y^3\right)\Big|_0^{2x} dx$$

$$= \int_0^1 \left(24x + 2x^3 - \frac{8}{3}x^3\right) dx$$

$$= \int_0^1 \left(24x - \frac{2}{3}x^3\right) dx$$

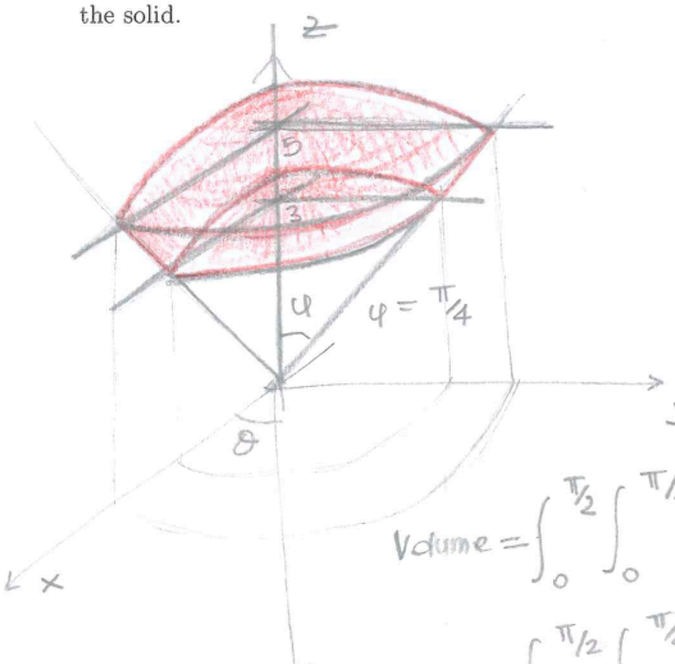
$$= 12x^2 - \frac{1}{6}x^4 \Big|_0^1$$

$$= 12 - \frac{1}{6}$$

$$= \frac{71}{6}$$

ANSWER: $\frac{71}{6}$

(5) (10 points) A solid occupies the region which is in the 1-st octant (where $x \geq 0$, $y \geq 0$, $z \geq 0$), inside the half-cone given by $z = \sqrt{x^2 + y^2}$, and between the spheres of radius 3 and 5 centered at the origin. Suppose that x, y, and z are measured in meters, and that the density function is $\delta(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$ kg/m^3. Calculate the mass of the solid.



Use spherical coordinates

$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$

$$dV = \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta$$

$$\text{Volume} = \int_0^{\pi/2} \int_0^{\pi/4} \int_3^5 \frac{\rho^2 \operatorname{sen}\varphi}{\rho^2} \, d\rho \, d\varphi \, d\theta =$$

$$= \int_0^{\pi/2} \int_0^{\pi/4} 2 \operatorname{sen}\varphi \, d\varphi \, d\theta$$

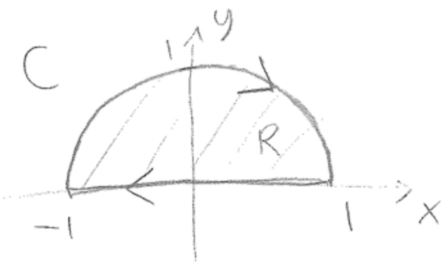
$$= \int_0^{\pi/2} -2\cos\varphi \Big|_0^{\pi/4} d\theta$$

$$= \int_0^{\pi/2} 2\left(1 - \frac{\sqrt{2}}{2}\right) d\theta$$

$$= \pi\left(1 - \frac{\sqrt{2}}{2}\right)$$

ANSWER: $\pi\left(1 - \frac{\sqrt{2}}{2}\right)$

(6) (12 points) Given $\vec{F}(x,y) = \big(\underbrace{\sin(x)e^{x^2} - 8y^3}_{P(x,y)}, \underbrace{8x^3 - y^4\ln(1+y^2)}_{Q(x,y)}\big)$, find the line integral of $\vec{F}$ along C, if C is the oriented curve that starts at $(-1,0)$ goes along the semi-circle of radius 1 centered at the origin with $y \geq 0$, and then goes back along the x-axis to $(-1,0)$.



R ... region bounded by C

→ polar coordinates:

$0 \leq r \leq 1$

$0 \leq \theta \leq \pi$

use Green's theorem

C oriented clockwise = negative orientation, so

$$\int_C \vec{F}\cdot d\vec{r} = -\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$\frac{\partial Q}{\partial x} = 24x^2, \quad \frac{\partial P}{\partial y} = -24y^2$$

So $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 24x^2 + 24y^2 = 24r^2$

$$\int_C \vec{F}\cdot d\vec{r} = -\iint_R 24r^2\, dA = -\int_0^{\pi}\int_0^{1} 24r^2\, r\, dr\, d\theta$$

$$= -\pi \cdot 24 \frac{r^4}{4}\Big|_0^1 = -6\pi$$

ANSWER: -6π

(7) (12 points) Given $\vec{F}(x,y) = (1 + ye^x + y, x + e^x)$, using the Fundamental Theorem of Line Integrals, find the line integral of $\vec{F}$ along C, if C is the oriented curve that consists of the line segment from $(0,0)$ to $(13,1)$, followed by the line segment from $(13,1)$ to $(1,1)$.

$P(x,y) = 1 + ye^x + y, \quad Q(x,y) = x + e^x$

$Q_x = 1 + e^x = P_y. \Rightarrow \vec{F}(x,y)$ is conservative.

If $\vec{F} = \nabla f$, then $f_x = 1 + ye^x + y \Rightarrow f = \int (1 + ye^x + y)\,dx = x + ye^x + xy + g(y)$

Now $x + e^x = f_y = e^x + x + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = c.$

So one potential of $\vec{F}(x,y)$ is $f(x,y) = x + ye^x + xy$

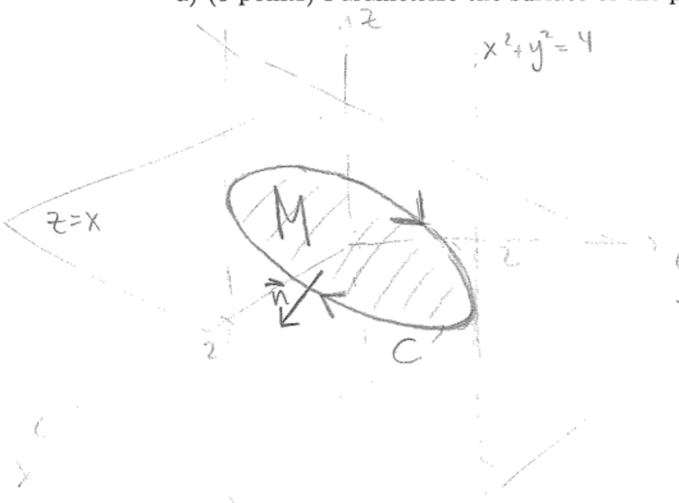
By Fundamental Theorem of Line Integrals,

$\int_C \vec{F} \cdot dr = f(1,1) - f(0,0) = (1 + e^1 + 1) - (0 + 0 + 0) = e + 2.$

ANSWER: $e+2$

(8) (12 points) Let C be the curve that is the intersection of the plane given by $z = x$ with the cylinder given by $x^2 + y^2 = 4$, oriented clockwise as viewed from above.

a) (5 points) Parametrize the surface of the plane $z = x$ that is bounded by the curve C.



M lies in the plane $z=x$ and inside the cylinder $x^2+y^2=4$

parameterization :

$\vec{r}(u,v) = (u\cos v,\ u\sin v,\ u\cos v)$

$D: \begin{cases} 0 \le u \le 2 \\ 0 \le v \le 2\pi \end{cases}$

ANSWER: ____________________

b) (7 points) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = (2z, x, 3)$.

$C = \partial M$ - boundary of M $\longrightarrow$ Stokes' thm: $\int_C \vec{F}\cdot d\vec{r} = \iint_M (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$

$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z & x & 3 \end{vmatrix} = \vec{i}(0) - \vec{j}(-2) + \vec{k}(1) = (0, \square 2, 1)$

$\vec{r}_u(u,v) = (\cos v, \sin v, \cos v)$

$\vec{r}_v(u,v) = (-u\sin v, u\cos v, -u\sin v)$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & \cos v \\ -u\sin v & u\cos v & -u\sin v \end{vmatrix} = \vec{i}(-u) - \vec{j}(0) + \vec{k}(u) = (-u, 0, u)$

M oriented such that normal has negative z-component $\rightarrow$ choose $(u, 0, -u)$ in integral

ANSWER: ____________________

$\iint_M (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = \iint_D (\vec{\nabla}\times\vec{F}(\vec{r}(u,v)))\cdot(\vec{r}_v\times\vec{r}_u)\,du\,dv$

$= \int_0^{2\pi}\int_0^2 \underbrace{(0,\square 2,1)\cdot(u,0,-u)}_{-u}\,du\,dv = -\int_0^{2\pi}\int_0^2 u\,du\,dv = -\int_0^{2\pi} \frac{u^2}{2}\Big|_0^2\,dv$

$= \boxed{-4\pi}$

(9) (12 points) Evaluate the flux integral $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS$ where

$$\vec{F} = (z^2 + xy^2, yx^2, y+z)$$

and S is the surface given by $z = x^2 + y^2$ that is above the disk of radius 1 in the xy-plane centered at the origin. The surface is oriented by the upward normal.

One method. See next page for another.

Parametrize S as $\vec{r}(u,v) = (u\cos v, u\sin v, u^2)$, $u\in[0,1]$, $v\in[0,2\pi]$.

$\vec{r}_u = (\cos v, \sin v, 2u)$, $\vec{r}_v = (-u\sin v, u\cos v, 0)$

$$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-2u^2\cos v, -2u^2\sin v, u)$$

Since $u\geq 0$, $\vec{r}_u\times\vec{r}_v$ points upward, in agreement with the orientation.

So $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$

$$= \int_0^{2\pi}\int_0^1 (u^4 + u^3\cos v\sin^2 v,\ u^3\sin v\cos^2 v,\ u\sin v + u^2)\cdot(-2u^2\cos v, -2u^2\sin v, u)\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 (-2u^6\cos v - 2u^5\sin^2 v\cos^2 v - 2u^5\sin^2 v\cos^2 v + u^2\sin v + u^3)\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 (-2u^6\cos v - 4u^5\sin^2 v\cos^2 v + u^2\sin v + u^3)\,du\,dv$$

$$= \int_0^{2\pi}\left(-\frac{2}{7}u^7\cos v - \frac{2}{3}u^6\sin^2 v\cos^2 v + \frac{1}{3}u^3\sin v + \frac{1}{4}u^4\Big|_{u=0}^{u=1}\right)dv$$

$$= \int_0^{2\pi}\left(-\frac{2}{7}\cos v - \frac{2}{3}\sin^2 v\cos^2 v + \frac{1}{3}\sin v + \frac{1}{4}\right)dv$$

$$= \int_0^{2\pi}\left(-\frac{2}{7}\cos v - \frac{1}{12}(1-\cos 4v) + \frac{1}{3}\sin v + \frac{1}{4}\right)dv$$

$$= -\frac{2}{7}\sin v + \frac{1}{48}\sin 4v - \frac{1}{3}\cos v + \frac{1}{6}v\Big|_0^{2\pi}$$

$$= \frac{\pi}{3}$$

ANSWER: $\frac{\pi}{3}$

(9) (12 points) Evaluate the flux integral $\iint_S \vec{F}\cdot\vec{n}\,dS$ where

$$\vec{F} = (z^2 + xy^2, yx^2, y + z)$$

and S is the surface given by $z = x^2 + y^2$ that is above the disk of radius 1 in the xy-plane centered at the origin. The surface is oriented by the upward normal.

Let M be the closed surface formed by S and the disk $S' = \{(x, y, 1) \mid x^2+y^2 \le 1\}$ with downward orientation.

By divergence theorem, $-\iint_{\partial M} \vec{F}\cdot\vec{n}\,dS = \iiint_M \vec{\nabla}\cdot\vec{F}\,dV = \iiint_M (y^2+x^2+1)\,dV$

$= \int_0^{2\pi}\int_0^1\int_{r^2}^1 (r^2+1)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r(1+r^2)(1-r^2)\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 (r - r^5)\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2} - \frac{r^6}{6}\Big|_{r=0}^1 d\theta = \int_0^{2\pi} \frac{1}{3}\,d\theta = \frac{2\pi}{3}$.

S' can be parametrized as $\vec{r}(u,v) = (u\cos v, u\sin v, 1)$, $u \in [0,1]$, $v \in [0, 2\pi]$

$\vec{r}_u = (\cos v, \sin v, 0)$ $\Longrightarrow$ $\vec{r}_u \times \vec{r}_v = (0, 0, u)$ upward

$\vec{r}_v = (-u\sin v, u\cos v, 0)$ Multiply by -1.

$\iint_{S'} \vec{F}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^1 \vec{F}\cdot(\vec{r}_v \times \vec{r}_u)\,du\,dv = \int_0^{2\pi}\int_0^1 -u(u\sin v + 1)\,du\,dv$

$= \int_0^{2\pi} -\frac{u^3}{3}\sin v - \frac{u^2}{2}\Big|_{u=0}^1 dv = \int_0^{2\pi}\left(-\frac{1}{3}\sin v - \frac{1}{2}\right)dv = \frac{1}{3}\cos v - \frac{1}{2}v\Big|_0^{2\pi} = -\pi$.

Then $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{\partial M} \vec{F}\cdot\vec{n}\,dS - \iint_{S'} \vec{F}\cdot\vec{n}\,dS = -\frac{2\pi}{3} - (-\pi) = \frac{\pi}{3}$

ANSWER: $\frac{\pi}{3}$